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A Physicist's Formulary

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Contents

1 Introduction

This formulary is intended to provide some of the essential identities or theorems which are easily being forgotten if not used regularly. The hypothesis or validity conditions are omitted in all cases where they are supposed to be obvious enough in order that the reader already knows them, or is able to easily find them out. This formulary is primarily aimed at theoretical physics, therefore it doesn't contain many numerical values or considerations on units. The content is based on lectures given to undergraduate students in physics at the EPFL (1996–2001). The latest version may be downloaded from http://www.francoiscoppex.com, under "publications". If the reader needs more specific mathematical relations that cannot be found in this formulary, it might then be useful to look at I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (1994).

Note that in order to have this formulary in a very handy form you may try the command *psnup -4 physformulary.ps* > *p4.ps*, then print *p4.ps* on both sides, and finally fold the formulary in four.

2 Mathematics

2.1 Algebra

2.1.1 Trigonometric Identities

•*Expression of sin, cos, tg, ctg in function of the other trigonometric functions*

$\nearrow$	$\sin(x)$	$\cos(x)$	$\text{tg}(x)$	$\text{ctg}(x)$
$\sin(x)$		$\sqrt{1-\cos(x)^2}$	$\frac{\text{tg}(x)}{\sqrt{1+\text{tg}(x)^2}}$	$\frac{1}{\sqrt{1+\text{ctg}(x)^2}}$
$\cos(x)$	$\sqrt{1-\sin(x)^2}$		$\frac{1}{\sqrt{1+\text{tg}(x)^2}}$	$\frac{\text{ctg}(x)}{\sqrt{1+\text{ctg}(x)^2}}$
$\text{tg}(x)$	$\frac{\sin(x)}{\sqrt{1-\sin(x)^2}}$	$\frac{\sqrt{1-\cos(x)^2}}{\cos(x)}$		$\frac{1}{\text{ctg}(x)}$
$\text{ctg}(x)$	$\frac{\sqrt{1-\sin(x)^2}}{\sin(x)}$	$\frac{\cos(x)}{\sqrt{1-\cos(x)^2}}$	$\frac{1}{\text{tg}(x)}$	

•*Most common identities*

$$\cos(x)+\cos(y) = 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right); \qquad \cos(x)-\cos(y) = -2\sin\left(\tfrac{x-y}{2}\right)\sin\left(\tfrac{x+y}{2}\right)$$
$$\sin(x)\pm\sin(y) = 2\sin\left(\tfrac{x\pm y}{2}\right)\cos\left(\tfrac{x\mp y}{2}\right)$$

$$\sin(x)\sin(y) = \tfrac{1}{2}\left(-\cos(x+y)+\cos(x-y)\right); \qquad \sin(x)\cos(y) = \tfrac{1}{2}\left(\sin(x+y)+\sin(x-y)\right)$$
$$\cos(x)\sin(y) = \tfrac{1}{2}\left(\sin(x+y)-\sin(x-y)\right); \qquad \cos(x)\cos(y) = \tfrac{1}{2}\left(\cos(x+y)+\cos(x-y)\right)$$

$$\sin(x\pm y) = \sin(x)\cos(y)\pm\cos(x)\sin(y); \qquad \cos(x\pm y) = \cos(x)\cos(y)\mp\sin(x)\sin(y)$$
$$\text{tg}(x\pm y) = \frac{\text{tg}(x)\pm\text{tg}(y)}{1\mp\text{tg}(x)\,\text{tg}(y)}$$

•*Half angle*

$$\sin\left(\tfrac{x}{2}\right)^2 = \frac{1-\cos(x)}{2}; \qquad \cos\left(\tfrac{x}{2}\right)^2 = \frac{1+\cos(x)}{2}$$
$$\text{tg}\left(\tfrac{x}{2}\right)^2 = \frac{1-\cos(x)}{1+\cos(x)}; \qquad \text{tg}\left(\tfrac{x}{2}\right) = \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)}$$

•*Other Identities*

$$\cos(x) = \tfrac{1}{2}\left(e^{ix}+e^{-ix}\right); \qquad \sin(x) = \tfrac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$
$$\sin(x) = \frac{2\,\text{tg}\left(\frac{x}{2}\right)}{1+\text{tg}\left(\frac{x}{2}\right)^2}; \qquad \cos(x) = \frac{1-\text{tg}\left(\frac{x}{2}\right)^2}{1+\text{tg}\left(\frac{x}{2}\right)^2}$$
$$\text{tg}(x) = \frac{2\,\text{tg}\left(\frac{x}{2}\right)}{1-\text{tg}\left(\frac{x}{2}\right)^2}$$

2.1.2 Hyperbolic Identities

•*Sum and Difference of Angles*

$$\text{sh}(x\pm y) = \text{sh}(x)\,\text{ch}(y)\pm\text{ch}(x)\,\text{sh}(y); \qquad \text{ch}(x\pm y) = \text{ch}(x)\,\text{ch}(y)\pm\text{sh}(x)\,\text{sh}(y)$$
$$\text{th}(x\pm y) = \frac{\text{th}(x)\pm\text{th}(y)}{1\pm\text{th}(x)\,\text{th}(y)}$$

•*Multiples of an Angle*

$$\text{sh}(2x) = 2\,\text{sh}(x)\,\text{ch}(x); \qquad \text{sh}\left(\tfrac{x}{2}\right)^2 = \frac{\text{ch}(x)-1}{2}$$
$$\text{ch}(2x) = \text{sh}(x)^2+\text{ch}(x)^2; \qquad \text{ch}\left(\tfrac{x}{2}\right)^2 = \frac{\text{ch}(x)+1}{2}$$
$$\text{th}(2x) = \frac{2\,\text{th}(x)}{1+\text{th}(x)^2}; \qquad \text{th}\left(\tfrac{x}{2}\right)^2 = \frac{\text{ch}(x)-1}{\text{sh}(x)} = \frac{\text{sh}(x)}{\text{ch}(x)+1}$$

•*Other Identities*

$$\text{ch}(x) = \tfrac{1}{2}\left(e^{x}+e^{-x}\right); \qquad \text{sh}(x) = \tfrac{1}{2}\left(e^{x}-e^{-x}\right)$$
$$\text{ch}(x)^2-\text{sh}(x)^2 = 1; \qquad \text{th}(x)^2+\frac{1}{\text{ch}(x)^2} = 1$$

2.1.3 Products

$\prod_{k=1}^{\infty}\left(1-\left(\frac{x}{\pi k}\right)^2\right)=\prod_{k=1}^{\infty}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(x)}{x}$; $\prod_{k=1}^{\infty}\left(1-\frac{4x^2}{(2k-1)^2}\right)=\cos(\pi x)$

$\prod_{k=0}^{\infty}\left(1+x^{2^k}\right)=\frac{1}{1-x}$; $\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k+1}}{2k-1}\right)=\sqrt{2}$

2.1.4 Sums

$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{6}\pi^2$; $\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{1}{90}\pi^4$

$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{1}{8}\pi^2$; $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}=\frac{1}{96}\pi^4$

$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{1}{4}\pi$; $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}=\ln(2)$

$\sum_{k=1}^{n}k=\frac{1}{2}n(n+1)$; $\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1)$

$\sum_{k=1}^{n}k^3=\left(\frac{1}{2}n(n+1)\right)^2$; $\sum_{k=1}^{n}\left(\frac{1}{k^p}-\frac{1}{(k+1)^p}\right)=1-\frac{1}{(n+1)^p},\quad p\in\mathbb{R}$

$\sum_{k=0}^{n}x^k=\frac{1-x^{n+1}}{1-x}$; $\sum_{k=0}^{n}\binom{m+k}{m}=\binom{m+n+1}{m+1}$

$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}=(a+b)^n$

2.1.5 Convergence Criterions

•*Comparison Criteria*
Let $z_n\in\mathbb{C}$.
- $\sum_n z_n$ converges absolutely if $|z_n|\le\alpha_n$, $\forall n>n_0$, $\sum_n\alpha_n<\infty$
- $\sum_n z_n$ does not converge absolutely if $|z_{k_n}|\ge\beta_n\ge 0$, $k_i<k_j\,\forall i<j$, $\sum_n\beta_n<\infty$

•*Geometrical Sum Criteria*
$\sum_n z^n$ converges absolutely $\iff$ $|z|<1$ and in this case $\sum_n z^n=\frac{1}{1-z}$

Cauchy's Criteria
Let $L=\limsup_{n\to\infty}|z_n|^{1/n}$, $z_n\in\mathbb{C}$.
- $L<1\Rightarrow\sum_n z_n$ converges, $\sum_n|z_n|<\infty$
- $L>1\Rightarrow\sum_n z_n$ diverges

•*d'Alembert's Criteria*
Let $L=\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right|$, $z_n\ne 0\,\forall n>n_0$, $n_0\ne 0$.
- $L<1\Rightarrow\sum_n z_n$ converges, $\sum_n|z_n|<\infty$
- $L>1\Rightarrow\sum_n z_n$ diverges

•*d'Abel's Criteria*
Suppose that $\sum_n a_n$ converges, and $\{b_n\}$ is a monotonous bounded sequence, then $\sum_n a_nb_n$ converges.

•*Leibnitz's Criteria*
Suppose that $\{a_n\}$ is a monotonous sequence of real numbers so that $\lim_{n\to\infty}a_n=0$, then $\sum_n(-1)^na_n$ converges.

2.1.6 Linear Algebra

•*Determinants*
Let $\boldsymbol{A}=\{a_{ij}\}_{i,j=1}^{n}\in\mathbb{C}^{n\times n}$; $\boldsymbol{B}\in\mathbb{C}^{n\times n}$; $\boldsymbol{A}[i,j]$ the matrix obtained from $\boldsymbol{A}$ by erasing the line i and column j.
Definition

$$\det(\boldsymbol{A})=\sum_{p_i\in\mathcal{S}_n}^{n!}\operatorname{sign}(p_i)a_{1p_i(1)}\cdots a_{np_i(n)} \tag{2.1}$$

Properties

$$\det(\boldsymbol{A}) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(\boldsymbol{A}[i,j]) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(\boldsymbol{A}[i,j]) \tag{2.2}$$

$$\det(\boldsymbol{A}\cdot\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) \tag{2.3}$$

$$\det\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B} & \boldsymbol{A}\end{pmatrix} = \det(\boldsymbol{A}+\boldsymbol{B})\det(\boldsymbol{A}-\boldsymbol{B}) \tag{2.4}$$

•*Comatrix (Cramer's Rule)*

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\tilde{\boldsymbol{A}}, \qquad \tilde{\boldsymbol{A}} = \{\tilde{a}_{ij}\}_{i,j=1}^{n} = \left\{(-1)^{i+j}\det(\boldsymbol{A}[j,i])\right\}_{i,j=1}^{n} \tag{2.5}$$

•*Diagonalization*
Let $\boldsymbol{A},\boldsymbol{D},\boldsymbol{S}\in\mathbb{C}^{n\times n}$; $\lambda_i\in\mathbb{C}$; $\mathbf{V}_{\lambda_i}\in\mathbb{C}^{n\times 1}$; $\boldsymbol{A}\cdot\mathbf{V}_{\lambda_i}=\lambda_i\mathbf{V}_{\lambda_i}$; $i=1,\ldots,n$; $\boldsymbol{D}_{ij}=\lambda_i\delta_{ij}$; $\boldsymbol{D}=\boldsymbol{S}^{-1}\cdot\boldsymbol{A}\cdot\boldsymbol{S}$; $\boldsymbol{S}=(\mathbf{V}_{\lambda_1}|\cdots|\mathbf{V}_{\lambda_n})$, then
- $\boldsymbol{A}\in\mathbb{R}^{n\times n}$; $\boldsymbol{A}=\boldsymbol{A}^t \Longrightarrow \exists\boldsymbol{S}$; $\boldsymbol{S}^{-1}=\boldsymbol{S}^t$
- $\boldsymbol{A}\in\mathbb{C}^{n\times n}$; $\boldsymbol{A}=\boldsymbol{A}^\star \Longrightarrow \exists\boldsymbol{S}$; $\boldsymbol{S}^{-1}=\boldsymbol{S}^\star$; $\lambda_i\in\mathbb{R}$, $i=1,\ldots,n$
- $\boldsymbol{A}\in\mathbb{C}^{n\times n}$; $\boldsymbol{A}\cdot\boldsymbol{A}^\star=\mathbb{1} \Longrightarrow \exists\boldsymbol{S}$; $\boldsymbol{S}^{-1}=\boldsymbol{S}^\star$; $|\lambda_i|=1$, $i=1,\ldots,n$

2.1.7 Rotation Matrix

Let $\{\hat{\mathbf{e}}_i\}_{i=1}^{3}$ be the cartesian orthonormal base; $\boldsymbol{R}_{\hat{\mathbf{e}}_i}$ the rotation matrix around $\hat{\mathbf{e}}_i$.

$$\boldsymbol{R}_{\hat{\mathbf{e}}_1} = \begin{pmatrix} 1 & 0 & 0\\ 1 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}; \quad \boldsymbol{R}_{\hat{\mathbf{e}}_2} = \begin{pmatrix} \cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}; \quad \boldsymbol{R}_{\hat{\mathbf{e}}_3} = \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

2.1.8 Levi-Civita Symbol

$$\varepsilon_{\mu_1\mu_2\ldots\mu_n} = \begin{cases} \mathrm{sign}(p), & \text{if } \mu_1\mu_2\ldots\mu_n = p(1\,2\ldots n),\ p\in\mathcal{S}_n\\ 0, & \text{otherwise}\end{cases} \tag{2.6}$$

2.2 One Variable Real Analysis

2.2.1 Taylor Sums

$$\begin{aligned}
\exp(x) &= \textstyle\sum_{k=0}^{\infty}\frac{x^k}{k!} &&\simeq 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}\\
\ln(1+x) &= \textstyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} &&\simeq x-\frac{x^2}{2}+\ldots+(-1)^{n+1}\frac{x^n}{n}
\end{aligned}$$

$$\begin{aligned}
\cos(x) &= \textstyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} &&\simeq 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}\\
\sin(x) &= \textstyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} &&\simeq x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}\\
\mathrm{asin}(x) &= x+\textstyle\sum_{k=1}^{\infty}\frac{(2k-1)!!}{2^k k!}\frac{x^{2k+1}}{2k+1} &&\simeq x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots+\frac{(2n-1)!!}{2^n n!}\frac{x^{2n+1}}{2n+1}\\
\mathrm{atg}(x) &= \textstyle\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1} &&\simeq x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots+(-1)^n\frac{x^{2n+1}}{2n+1}
\end{aligned}$$

$$\begin{aligned}
\mathrm{ch}(x) &= \textstyle\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} &&\simeq 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}\\
\mathrm{sh}(x) &= \textstyle\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} &&\simeq x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+\frac{x^{2n+1}}{(2n+1)!}\\
\mathrm{ash}(x) &= x+\textstyle\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k k!}\frac{x^{2k+1}}{2k+1} &&\simeq x-\frac{x^3}{6}+\frac{3x^5}{40}+\ldots+(-1)^n\frac{(2n-1)!!}{2^n n!}\frac{x^{2n+1}}{2n+1}\\
\mathrm{ath}(x) &= \textstyle\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1} &&\simeq x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n+1}}{2n+1}
\end{aligned}$$

$\frac{1}{1+x}$	$=$	$\sum_{k=0}^{\infty}(-1)^k x^k$	$\simeq$	$1-x+x^2+\ldots+(-1)^n x^n$
$\frac{1}{(1+x)^2}$	$=$	$\sum_{k=0}^{\infty}(-1)^k (k+1) x^k$	$\simeq$	$1-2x+3x^2+\ldots+(-1)^n (n+1) x^n$
$\sqrt{1+x}$	$=$	$1+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2k-1)!!}{2^k k!}x^k$	$\simeq$	$1+\frac{x}{2}-\frac{x^2}{8}+\ldots+(-1)^{n+1}\frac{(2n-3)!!}{2^n n!}x^n$
$\frac{1}{\sqrt{1+x}}$	$=$	$1+\sum_{k=1}^{\infty}(-1)^{k}\frac{(2k-1)!!}{2^k k!}x^k$	$\simeq$	$1-\frac{x}{2}+\frac{3x^2}{8}+\ldots+(-1)^{n}\frac{(2n-1)!!}{2^n n!}x^n$
$(1+x)^\alpha$	$=$	$1+\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}(\alpha-i)}{k!}x^k$	$\simeq$	$1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\ldots+\frac{\prod_{i=0}^{n-1}(\alpha-i)}{n!}x^n$
$\ln\left(\frac{1+x}{1-x}\right)$	$=$	$2\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$	$\simeq$	$2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n+1}}{2n+1}\right)$
$\int_0^x \mathrm{d}y\, \mathrm{e}^{-y^2}$	$=$	$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{k!(2k+1)}$	$\simeq$	$x-\frac{x^3}{3}+\frac{x^5}{2!5}-\frac{x^7}{3!7}+\ldots+(-1)^n\frac{x^{2n+1}}{n!(2n+1)}$

2.2.2 Integrals

•*Gaussian Integrals*
Let $\alpha,\beta,\gamma\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$; $\mathbf{x}$, $\mathbf{a}\in\mathbb{R}^d$; $\boldsymbol{A}:\mathbb{R}^d\to\mathbb{R}^d$ an operator (matrix); n_p (n_n) the number of positive (negative) eigenvalues of $\boldsymbol{A}$, $\mathrm{sign}(\boldsymbol{A})=n_p-n_n$.

$$\int_{\mathbb{R}^d}\mathrm{d}^d\mathbf{x}\,\exp\left(-\alpha\mathbf{x}^2+\beta\mathbf{x}+\gamma\right)=\left(\frac{\pi}{\alpha}\right)^{d/2}\exp\left(\frac{4\alpha\gamma+\beta^2}{4\alpha}\right)\tag{2.7}$$

$$\int_{\mathbb{R}^d}\mathrm{d}^d\mathbf{x}\,\exp\left(-\tfrac{1}{2}\langle\mathbf{x}|\boldsymbol{A}\mathbf{x}\rangle+\langle\mathbf{a}|\mathbf{x}\rangle\right)=\frac{(2\pi)^{d/2}}{\sqrt{\det(\boldsymbol{A})}}\exp\left(\tfrac{1}{2}\langle\mathbf{a}|\boldsymbol{A}^{-1}\mathbf{a}\rangle\right)\tag{2.8}$$

$$\int_{\mathbb{R}^d}\mathrm{d}^d\mathbf{x}\,\exp\left(\tfrac{i}{2}\langle\mathbf{x}|\boldsymbol{A}\mathbf{x}\rangle\right)=\frac{(2\pi)^{d/2}}{\sqrt{\det(\boldsymbol{A})}}\exp\left(i\frac{\pi}{4}\,\mathrm{sign}(\boldsymbol{A})\right)\tag{2.9}$$

•*Integrals of Gaussian Moments*

$$\int_{\mathbb{R}^d}\mathrm{d}^d\mathbf{x}\,|\mathbf{x}|^n\,\mathrm{e}^{-\alpha\mathbf{x}^2}=\frac{\pi^{d/2}}{\alpha^{\frac{d+n}{2}}}\frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\tag{2.10}$$

$$\int_{\mathbb{R}^d}\mathrm{d}^d\mathbf{x}\,|\mathbf{x}|^n\,\mathrm{e}^{-\alpha\mathbf{x}^2}x_ix_j=\frac{\pi^{d/2}}{\alpha^{(d+n+2)/2}}\frac{d+n}{2d}\frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\delta_{ij}\tag{2.11}$$

$$\begin{aligned}\int_{\mathbb{R}^d}\mathrm{d}^d\mathbf{x}\,|\mathbf{x}|^n\,\mathrm{e}^{-\alpha\mathbf{x}^2}x_ix_jx_kx_l=&\frac{\pi^{d/2}}{\alpha^{(d+n+4)/2}}\frac{3}{4}\frac{(d+n)(d+n+2)}{d(d+2)}\frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\\&\times\left\{\delta_{ijkl}+\frac{1}{3}\left[\delta_{ij}\delta_{kl}(1-\delta_{ik})+\delta_{ik}\delta_{jl}(1-\delta_{ij})+\delta_{il}\delta_{jk}(1-\delta_{ij})\right]\right\}\end{aligned}\tag{2.12}$$

•*Exponential-like Integrals*

$$\int_0^\infty\mathrm{d}x\,x^n\mathrm{e}^{-\alpha x}=\frac{n!}{\alpha^{n+1}}\tag{2.13}$$

•*Angular integrals*
Let $\mathbf{g}$, $\widehat{\boldsymbol{\sigma}}=(\sigma_1,\ldots,\sigma_d)\in\mathbb{R}^d$, $|\widehat{\boldsymbol{\sigma}}|=1$. We adopt the notation $\int\mathrm{d}\widehat{\boldsymbol{\sigma}}=\int_{|\mathbf{x}|=1}\mathrm{d}^d\mathbf{x}$. In the integrals below, the results when $\theta(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})$ is absent are obtained upon multiplying the value of β_n by two.

$$\int\mathrm{d}\widehat{\boldsymbol{\sigma}}\,\theta(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})^n\sigma_i\sigma_j=\frac{\beta_n}{n+d}g^{n-2}(ng_ig_j+g^2\delta_{ij})\tag{2.14}$$

$$\int\mathrm{d}\widehat{\boldsymbol{\sigma}}\,\theta(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\widehat{\boldsymbol{\sigma}}\cdot\mathbf{g})^n\sigma_i=\beta_{n+1}g^{n-1}g_i\tag{2.15}$$

$$\beta_n=\int\mathrm{d}\widehat{\boldsymbol{\sigma}}\,\theta(\widehat{\boldsymbol{\sigma}}\cdot\widehat{\mathbf{g}})(\widehat{\boldsymbol{\sigma}}\cdot\widehat{\mathbf{g}})^n=\pi^{(d-1)/2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)}\tag{2.16}$$

• *Hypersphere Integrals*

$$\int_{|\mathbf{x}|\le R} \mathrm{d}^n\mathbf{x} = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n \tag{2.17}$$

$$\int_{|\mathbf{x}|=R} \mathrm{d}^n\mathbf{x} = \frac{2\pi^{n/2}}{\Gamma(n/2)} R^{n-1} \tag{2.18}$$

• *Trigonometric Functions*

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{\cos^2 x} = \int \mathrm{d}x\ \mathrm{tg}\, x = -\ln(|\cos x|) \tag{2.19}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-1}{\sin^2 x} = \int \mathrm{d}x\ \mathrm{ctg}\, x = \ln(|\sin x|) \tag{2.20}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-\cos x}{\sin^2 x} = \int \mathrm{d}x\, \frac{1}{\sin x} = \ln\left(\left|\mathrm{tg}\left(\frac{x}{2}\right)\right|\right) \tag{2.21}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{\sin x}{\cos^2 x} = \int \mathrm{d}x\, \frac{1}{\cos x} = \ln\left(\left|\mathrm{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right) \tag{2.22}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}} = \int \mathrm{d}x\ \mathrm{asin}\, x = x\,\mathrm{asin}\, x + \sqrt{1-x^2} \tag{2.23}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-1}{\sqrt{1-x^2}} = \int \mathrm{d}x\ \mathrm{acos}\, x = x\,\mathrm{acos}\, x - \sqrt{1-x^2} \tag{2.24}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{1+x^2} = \int \mathrm{d}x\ \mathrm{atg}\, x = x\,\mathrm{atg}\, x - \ln\left(\sqrt{1+x^2}\right) \tag{2.25}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-1}{1+x^2} = \int \mathrm{d}x\ \mathrm{actg}\, x = x\,\mathrm{actg}\, x + \ln\left(\sqrt{1+x^2}\right) \tag{2.26}$$

• *Hyperbolic Functions*

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{\mathrm{ch}^2 x} = \int \mathrm{d}x\ \mathrm{th}\, x = \ln(\mathrm{ch}\, x) \tag{2.27}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-1}{\mathrm{sh}^2 x} = \int \mathrm{d}x\ \mathrm{cth}\, x = \ln(|\mathrm{sh}\, x|) \tag{2.28}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{\sqrt{1+x^2}} = \int \mathrm{d}x\ \mathrm{ash}\, x = x\,\mathrm{ash}\, x - \sqrt{x^2+1} \tag{2.29}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{\sqrt{x^2-1}} = \int \mathrm{d}x\ \mathrm{ach}\, x = x\,\mathrm{ach}\, x - \sqrt{x^2-1} \tag{2.30}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{1}{1-x^2} = \int \mathrm{d}x\ \mathrm{ath}\, x = x\,\mathrm{ath}\, x + \ln\left(\sqrt{1-x^2}\right) \tag{2.31}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-1}{1-x^2} = \int \mathrm{d}x\ \mathrm{acth}\, x = x\,\mathrm{acth}\, x + \ln\left(\sqrt{x^2-1}\right) \tag{2.32}$$

• *Square Roots*

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{x}{\sqrt{x^2 \pm a^2}} = \int \mathrm{d}x\, \sqrt{x^2 \pm a^2} = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left(x + \sqrt{x^2 \pm a^2}\right) \tag{2.33}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-x}{\sqrt{a^2 - x^2}} = \int \mathrm{d}x\, \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\,\mathrm{asin}\left(\frac{x}{a}\right) \tag{2.34}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{-x}{\sqrt{(x^2 \pm a^2)^3}} = \int \mathrm{d}x\, \frac{1}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right) \tag{2.35}$$

$$\int \mathrm{d}x \int \mathrm{d}x\, \frac{x}{\sqrt{(x^2 - a^2)^3}} = \int \mathrm{d}x\, \frac{1}{\sqrt{a^2 - x^2}} = \mathrm{asin}\left(\frac{x}{a}\right) \tag{2.36}$$

•*Polynomial Fractions*

$$\int \mathrm{d}x\,\frac{x}{(x^2\pm a)^n}=\frac{1}{2}\frac{(x^2\pm a)^{1-n}}{1-n} \tag{2.37}$$

$$\int \mathrm{d}x\,\frac{x^{p-1}}{x^p+a}=\tfrac{1}{p}\ln\left(|x^p+a|\right) \tag{2.38}$$

$$\int \mathrm{d}x\,\frac{x}{(x^2+ax+b)^2}=\frac{ax+2b}{(x^2+ax+b)(a^2-4b)}-2a\frac{\operatorname{atg}\left(\frac{2x+a}{\sqrt{4b-a^2}}\right)}{(4b-a^2)^{3/2}} \tag{2.39}$$

$$\int \mathrm{d}x\,\frac{ax+b}{x^2+2cx+s}=\frac{a}{2}\ln\left(|x^2+2cx+s|\right)+\frac{b-ca}{\sqrt{d-c^2}}\operatorname{atg}\left(\frac{x+c}{\sqrt{d-c^2}}\right) \tag{2.40}$$

$$\int \mathrm{d}x\int \mathrm{d}x\,\frac{-2(x+a)}{((x+a)^2+b^2)^2}=\int \mathrm{d}x\,\frac{1}{(x+a)^2+b^2}=\frac{1}{b}\operatorname{atg}\left(\frac{x+a}{b}\right) \tag{2.41}$$

$$\int \mathrm{d}x\int \mathrm{d}x\,\frac{\pm 2x}{(\mp x^2\pm a^2)^2}=\int \mathrm{d}x\,\frac{1}{\mp x^2\pm a^2}=\frac{1}{2a}\ln\left(\left|\frac{x\pm a}{\mp x+a}\right|\right) \tag{2.42}$$

•*Derivative of an Integral*

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_{h(y)}^{g(y)}\mathrm{d}x\,f(x,k(y))=g^{(1)}(y)f(g(y),k(y))-h^{(1)}(y)f(h(y),k(y))+\int_{h(y)}^{g(y)}\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}y}f(x,k(y)) \tag{2.43}$$

•*Spherical Change of Variables in* $\mathbb{R}^n$

Let $r>0$, $\varphi\in]0,2\pi[$, $\theta_i\in]0,\pi[\ \forall i=1,\ldots,n-2$, then the spherical change of variables in $\mathbb{R}^n$ is

$$\left\{\begin{array}{rcl} x_1 &=& r\sin\theta_1\ldots\sin\theta_{n-2}\cos\varphi \\ x_2 &=& r\sin\theta_1\ldots\sin\theta_{n-2}\sin\varphi \\ x_k &=& r\cos\theta_{k-2}\sin\theta_{k-1}\ldots\sin\theta_{n-2}, \quad k=3,\ldots,n-1 \\ x_n &=& r\cos\theta_{n-2} \end{array}\right. \tag{2.44}$$

with the Jacobian

$$J=r^{n-1}\prod_{k=1}^{n-2}(\sin\theta_k)^k\,. \tag{2.45}$$

2.2.3 Inequalities

Let $\|f\|_p=\left(\int_a^b \mathrm{d}x\,|f(x)|^p\right)^{1/p}$.

$$\int_a^b \mathrm{d}x\,|f(x)g(x)|\le\|f\|_p\|g\|_q,\qquad 1/p+1/q=1\ \text{(Hölder)} \tag{2.46}$$

$$\|f+g\|_p\le\|f\|_p+\|g\|_p,\qquad 1\le p<\infty\ \text{(Minkowski)} \tag{2.47}$$

$$|\langle f|g\rangle|\le\|f\|\,\|g\|,\qquad \text{(Cauchy-Schwartz)} \tag{2.48}$$

$$\|f\|\,\|g\|\le\alpha\|f\|^2+\frac{1}{4\alpha}\|g\|^2,\qquad \alpha>0\ \text{(Young)} \tag{2.49}$$

$$|\langle f|g\rangle|\le|f(a)|\max_{a\le\xi\le b}\left|\int_a^\xi \mathrm{d}x\,g(x)\right|,\qquad f(a)f(b)\ge 0,|f(a)|\ge|f(b)|\ \text{(Ostrowski)} \tag{2.50}$$

2.2.4 Dirac Distribution

$$\forall f(x) \text{ so that } \int_{\mathbb{R}} \mathrm{d}x\, f(x) = 1, \text{ then: } \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} f\left(\frac{x - x_0}{\varepsilon}\right) = \delta(x - x_0) \tag{2.51}$$

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}k\, \mathrm{e}^{ik(x - x_0)} \tag{2.52}$$

$$\delta(x - x_0) = \frac{d}{dx} \theta(x - x_0) \tag{2.53}$$

$$\int_{\mathbb{R}} \mathrm{d}x\, \varphi(x) \delta^{(n)}(x - x_0) = (-1)^n \varphi^{(n)}(x_0) \tag{2.54}$$

$$\delta(ax) = \tfrac{1}{|a|} \delta(x), \quad a \in \mathbb{R} \tag{2.55}$$

$$\delta\left(g(x)\right) = \sum_{i=1}^{n} \frac{1}{\left|g^{(1)}(x_i)\right|} \delta(x - x_i), \qquad \left\{g(x_i) = 0, g^{(1)}(x_i) \neq 0\right\}_{i=1}^{n} \tag{2.56}$$

$$\boldsymbol{\nabla}^2 \left(\tfrac{1}{r}\right) = -4\pi \delta\left(\mathbf{r}\right); \qquad \boldsymbol{\nabla} \cdot \left(\tfrac{\mathbf{r}}{r^3}\right) = 4\pi \delta(\mathbf{r}); \qquad \left(\boldsymbol{\nabla}^2 + \mathbf{k}^2\right) \tfrac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}}{r} = -4\pi\delta(\mathbf{r})$$

2.2.5 Dominated Convergence Theorem

Let $f_n(x)_{n \geq 1} \in \mathcal{L}^1(\mu)$; $\lim_{n \to \infty} f_n(x) = f(x)$; $|f_n(x)| \leq M(X) \in \mathcal{L}^1(\mu)$ $\forall x \in \Omega \subset \mathbb{R}$, $\forall n$.

$$\lim_{n \to \infty} \int_{\Omega} \mathrm{d}\mu\, f_n = \int_{\Omega} \mathrm{d}\mu\, f \tag{2.57}$$

2.3 Vector Analysis

Notation: f, g are scalar functions; $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are vectors; $\boldsymbol{T}$ is a tensor; $\{\hat{\mathbf{e}}_i\}_{i=1}^{3}$ is an orthonormal basis of $\mathbb{R}^3$; $\mathbf{r} = \sum_{i=1}^{3} x_i \hat{\mathbf{e}}_i$; $r = |\mathbf{r}|$; $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$.

2.3.1 Vector Identities

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} \tag{2.58}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\, \mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\, \mathbf{C} \tag{2.59}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0 \tag{2.60}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})\, (\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})\, (\mathbf{B} \cdot \mathbf{C}) \tag{2.61}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\, \mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\, \mathbf{D} \tag{2.62}$$

$$\boldsymbol{\nabla}(fg) = \boldsymbol{\nabla}(gf) = f \boldsymbol{\nabla} g + g \boldsymbol{\nabla} f \tag{2.63}$$

$$\boldsymbol{\nabla} \cdot (f\mathbf{A}) = f \boldsymbol{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\nabla} f \tag{2.64}$$

$$\boldsymbol{\nabla} \times (f\mathbf{A}) = f \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} f \times \mathbf{A} \tag{2.65}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B} \tag{2.66}$$

$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\, (\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}\, (\boldsymbol{\nabla} \cdot \mathbf{A}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\, \mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\, \mathbf{B} \tag{2.67}$$

$$\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) = (\boldsymbol{\nabla} \mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\, \mathbf{B} \tag{2.68}$$

$$\boldsymbol{\nabla}\, (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\, \mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\, \mathbf{A} \tag{2.69}$$

$$\boldsymbol{\nabla}^2 \mathbf{A} = \boldsymbol{\nabla}\, (\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} \tag{2.70}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0 \tag{2.71}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0 \tag{2.72}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{A} = 0 \tag{2.73}$$

$$\boldsymbol{\nabla} r = \hat{\mathbf{r}} \tag{2.74}$$

2.3.2 Differential Operators in Curvilinear Coordinates

•*Cylindrical Coordinates*
Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{2.75}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \qquad (\nabla f)_\phi = \frac{1}{r}\frac{\partial f}{\partial \phi}; \qquad (\nabla f)_z = \frac{\partial f}{\partial z} \tag{2.76}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial \phi} \tag{2.77}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \tag{2.78}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \tag{2.79}$$

Laplacian of a scalar function

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.80}$$

Laplacian of a vector

$$\left(\nabla^2 \mathbf{A}\right)_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2} \tag{2.81}$$

$$\left(\nabla^2 \mathbf{A}\right)_\phi = \nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2} \tag{2.82}$$

$$\left(\nabla^2 \mathbf{A}\right)_z = \nabla^2 A_z \tag{2.83}$$

Components of $(\mathbf{A} \cdot \nabla)\,\mathbf{B}$

$$((\mathbf{A} \cdot \nabla)\,\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_r}{\partial \phi} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r} \tag{2.84}$$

$$((\mathbf{A} \cdot \nabla)\,\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_\phi}{\partial \phi} + A_z\frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r} \tag{2.85}$$

$$((\mathbf{A} \cdot \nabla)\,\mathbf{B})_z = A_r\frac{\partial B_z}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_z}{\partial \phi} + A_z\frac{\partial B_z}{\partial z} \tag{2.86}$$

Divergence of a tensor

$$(\nabla \cdot \boldsymbol{T})_r = \frac{1}{r}\frac{\partial}{\partial r}(r\boldsymbol{T}_{rr}) + \frac{1}{r}\frac{\partial \boldsymbol{T}_{\phi r}}{\partial \phi} + \frac{\partial \boldsymbol{T}_{zr}}{\partial z} - \frac{\boldsymbol{T}_{\phi\phi}}{r} \tag{2.87}$$

$$(\nabla \cdot \boldsymbol{T})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\boldsymbol{T}_{r\phi}) + \frac{1}{r}\frac{\partial \boldsymbol{T}_{\phi\phi}}{\partial \phi} + \frac{\partial \boldsymbol{T}_{z\phi}}{\partial z} - \frac{\boldsymbol{T}_{\phi r}}{r} \tag{2.88}$$

$$(\nabla \cdot \boldsymbol{T})_z = \frac{1}{r}\frac{\partial}{\partial r}(r\boldsymbol{T}_{rz}) + \frac{1}{r}\frac{\partial \boldsymbol{T}_{\phi z}}{\partial \phi} + \frac{\partial \boldsymbol{T}_{zz}}{\partial z} \tag{2.89}$$

•*Spherical Coordinates*
Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{2.90}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \qquad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}; \qquad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \tag{2.91}$$

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} \tag{2.92}$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \tag{2.93}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta} \tag{2.94}$$

Laplacian of a scalar function

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{2.95}$$

Laplacian of a vector

$$\left(\nabla^2\mathbf{A}\right)_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\operatorname{ctg}\theta A_\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{2.96}$$

$$\left(\nabla^2\mathbf{A}\right)_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi} \tag{2.97}$$

$$\left(\nabla^2\mathbf{A}\right)_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi} \tag{2.98}$$

Components of $(\mathbf{A}\cdot\nabla)\,\mathbf{B}$

$$((\mathbf{A}\cdot\nabla)\,\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r} \tag{2.99}$$

$$((\mathbf{A}\cdot\nabla)\,\mathbf{B})_\theta = A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi} + \frac{A_\theta B_r}{r} - \frac{\operatorname{ctg}\theta A_\phi B_\phi}{r} \tag{2.100}$$

$$((\mathbf{A}\cdot\nabla)\,\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi} + \frac{A_\phi B_r}{r} + \frac{\operatorname{ctg}\theta A_\phi B_\theta}{r} \tag{2.101}$$

Divergence of a tensor

$$(\nabla\cdot\boldsymbol{T})_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\boldsymbol{T}_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\boldsymbol{T}_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial\boldsymbol{T}_{\phi r}}{\partial\phi} - \frac{\boldsymbol{T}_{\theta\theta}+\boldsymbol{T}_{\phi\phi}}{r} \tag{2.102}$$

$$(\nabla\cdot\boldsymbol{T})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\boldsymbol{T}_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\boldsymbol{T}_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial\boldsymbol{T}_{\phi\theta}}{\partial\phi} + \frac{\boldsymbol{T}_{\theta r}}{r} - \frac{\operatorname{ctg}\theta\boldsymbol{T}_{\phi\phi}}{r} \tag{2.103}$$

$$(\nabla\cdot\boldsymbol{T})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\boldsymbol{T}_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\boldsymbol{T}_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial\boldsymbol{T}_{\phi\phi}}{\partial\phi} + \frac{\boldsymbol{T}_{\phi r}}{r} - \frac{\operatorname{ctg}\theta\boldsymbol{T}_{\phi\theta}}{r} \tag{2.104}$$

2.3.3 Theorems (Green, Ostrogradsky, Stokes)

• *Green*
Let $\Omega \subset \mathbb{R}^2$ be a regular domain; $P, Q \in \mathrm{C}^1\left(\overline{\Omega}\right)$; $\mathrm{Fr}(\Omega)$ the frontier positively oriented of Ω.

$$\oint_{\mathrm{Fr}(\Omega)}\begin{pmatrix}P\\Q\end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}x\\\mathrm{d}y\end{pmatrix} = \int_\Omega \mathrm{d}x\,\mathrm{d}y\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \tag{2.105}$$

• *Ostrogradski (Divergence Theorem)*
Let $\Omega \subset \mathbb{R}^3$ be a regular domain parametrised by $\Omega = \psi(U)$; $U \subset \mathbb{R}^2$; $\mathbf{F} \in \mathrm{C}^1\left(\overline{\Omega}; \mathbb{R}^3\right)$; $\mathrm{Fr}(\Omega)$ the frontier positively oriented of Ω; $\boldsymbol{\nu}$ the unitary boundary continuous normal vector of Ω; $\mathrm{d}\sigma = \|D_1\psi \times D_2\psi\|\,\mathrm{d}x_1\mathrm{d}x_2$.

$$\oint_{\mathrm{Fr}(\Omega)}\mathrm{d}\sigma\,\mathbf{F}\cdot\boldsymbol{\nu} = \int_\Omega \mathrm{d}^3\mathbf{x}\,\nabla\cdot\mathbf{F} \tag{2.106}$$

•*Stokes in* $\mathbb{R}^3$
Let $S \subset \mathbb{R}^3$ be a regular surface; $\boldsymbol{\nu}$ its unitary boundary continuous normal vector; $\Omega \in S$ a regular domain delimited by a Jordan curve $\partial\Omega$; $\mathbf{F} \in \mathrm{C}^1\left(\overline{\Omega}; \mathbb{R}^3\right)$.

$$\int_{\partial\Omega} \mathrm{d}\mathbf{s} \cdot \mathbf{F} = \int_{\Omega} \mathrm{d}\sigma \, (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \boldsymbol{\nu} \tag{2.107}$$

2.4 Complex Analysis

2.4.1 Cauchy Formula

Let $f \in H(\Omega)$; $\Omega \subset \mathbb{C}$; let Ω be an open domain; $\boldsymbol{\gamma}(t)$ a simple closed piecewise C^1 curve in Ω, $t \in I \subset \mathbb{R}$, whose interior in contained in Ω; $\int_{\boldsymbol{\gamma}} \mathrm{d}\mathbf{s} \cdot \mathbf{F} = \int_I \mathrm{d}t \, \mathbf{F}\left(\boldsymbol{\gamma}(t)\right) \cdot \boldsymbol{\gamma}^{(1)}(t)$.

$$f(z) = \frac{1}{2\pi i} \oint_{\boldsymbol{\gamma}} \mathrm{d}\omega \, \frac{f(\omega)}{\omega - z} \tag{2.108}$$

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\boldsymbol{\gamma}} \mathrm{d}\omega \, \frac{f(\omega)}{(\omega - z)^{k+1}} \tag{2.109}$$

2.4.2 Laurent Sums

Let $f \in H(\mathrm{C})$; $\mathrm{C} = \{z : r_1 < |z - z_0| < r_2\}$, $0 \leq r_1 < r_2 \leq \infty$; let $\boldsymbol{\gamma}$ be a simple closed piecewise curve in a domain D so that $\overline{\mathrm{B}}(z_0; r_1) \subset \mathrm{D} \subset \overline{\mathrm{D}} \subset \overline{\mathrm{B}}(z_0; r_2)$.

$$f(z) = \sum_{k=0}^{\infty} a_k \left(z - z_0\right)^k + \sum_{k=1}^{\infty} b_k \left(z - z_0\right)^{-k} \tag{2.110}$$

$$a_k = \frac{1}{2\pi i} \oint_{\boldsymbol{\gamma}} \mathrm{d}z \, \frac{f(z)}{(z - z_0)^{-k-1}}, \quad k \in \mathbb{Z} \tag{2.111}$$

$$b_k = a_{-k} \tag{2.112}$$

2.4.3 Residues

It is said that $f(z)$ has a pole of order m at $z = z_0$ if $b_m \neq 0$ and $b_j = 0 \ \forall \ j > m$. In this case the residue of $f(z)$ is b_1.
•*First order pole*
- $b_1 = \lim_{z \to z_0} (z - z_0) f(z)$
- If $f(z) = \frac{A(z)}{B(z)}$, $B(z_0) = 0$, $B^{(1)}(z_0) \neq 0$, then $b_1 = \frac{A(z_0)}{B^{(1)}(z_0)}$

•*Pole of order m*
- Let $g(z) = (z - z_0)^m f(z)$, then $b_1 = \frac{g^{(m-1)}(z_0)}{(m-1)!}$
- In the general case $b_1 = \frac{1}{2\pi i} \oint_{\boldsymbol{\gamma}} \mathrm{d}z \, f(z)$, $z_0 \in \mathrm{D}_{\mathrm{int}}(\boldsymbol{\gamma})$

2.5 Ordinary Differential Equations

2.5.1 First Order Linear ODE

Let $\alpha(t) = \int_{t_0}^{t} \mathrm{d}s \, a(s)$.

$$\dot{x}(t) = a(t)x(t) + b(t) \iff x(t) = \mathrm{e}^{\alpha(t)} \left(x_0 + \int_{t_0}^{t} \mathrm{d}s \, b(s) \mathrm{e}^{-\alpha(s)} \right) \tag{2.113}$$

2.5.2 First Order Linear ODE System

Let $\mathbf{x}(t) \in \mathbb{R}^n$; $\boldsymbol{A} \in \mathbb{R}^{n \times n}$; $\mathbf{b}(t) \in \mathbb{R}^n$; $b_i \in \mathrm{C}^0(\mathbb{R}) \, \forall i = 1, \ldots, n$.

$$\dot{\mathbf{x}}(t) = \boldsymbol{A} \cdot \mathbf{x}(t) + \mathbf{b}(t) \iff \mathbf{x}(t) = \mathrm{e}^{(t - t_0)\boldsymbol{A}} \cdot \mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{d}s \ \mathrm{e}^{(t-s)\boldsymbol{A}} \cdot \mathbf{b}(s) \tag{2.114}$$

2.5.3 Bernoulli Equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(t)^n \iff x(t) = \frac{x_0 \mathrm{e}^{\alpha(t)}}{\left(1 + (1-n)x_0 \int_{t_0}^{t} \mathrm{d}s\, b(s)\mathrm{e}^{(n-1)\alpha(s)}\right)^{\frac{1}{n-1}}} \tag{2.115}$$

2.5.4 Second Order Linear ODE with Constant Coefficients

$$f^{(2)}(x) + af^{(1)}(x) + bf(x) = h(t) \tag{2.116}$$

Case 1: $a^2 - 4b > 0$

$$\lambda_\pm = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{2.117}$$

$$f(x) = c_1 \mathrm{e}^{\lambda_+ x} + c_2 \mathrm{e}^{\lambda_- x} + \int_0^x \mathrm{d}s\, \frac{\mathrm{e}^{\lambda_+(x-s)} - \mathrm{e}^{\lambda_-(x-s)}}{\lambda_+ - \lambda_-} h(s) \tag{2.118}$$

Case 2: $a^2 - 4b = 0$

$$f(x) = \mathrm{e}^{-\frac{a}{2}x}(c_1 + c_2 x) + \int_0^x \mathrm{d}s\,(x-s)\mathrm{e}^{-\frac{a}{2}(x-s)} h(s) \tag{2.119}$$

Case 3: $a^2 - 4b < 0$

$$\lambda = \frac{\sqrt{4b - a^2}}{2} \tag{2.120}$$

$$f(x) = \mathrm{e}^{-\frac{a}{2}x}(c_1 \cos(\lambda x) + c_2 \sin(\lambda x)) + \frac{1}{\lambda}\int_0^x \mathrm{d}s\, \mathrm{e}^{-\frac{a}{2}(x-s)} \sin(\lambda(x-s))h(s) \tag{2.121}$$

2.5.5 Eulerian Equation

Let $\{a_i\}_{i=1}^3 \in \mathbb{R}$, $a_2 \neq 0$, $b_0 = \frac{a_0}{a_2}$, $b_1 = \frac{a_1}{a_2} - 1$, $g(x) = f(\mathrm{e}^x)$.

$$a_2 x^2 f^{(2)}(x) + a_1 x f^{(1)}(x) + a_0 f(x) = 0 \iff g^{(2)}(x) + b_1 g^{(1)}(x) + b_0 g(x) = 0 \tag{2.122}$$

$$f(x) = \begin{cases} g(\ln(x)), & x > 0 \\ g(\ln(-x)), & x < 0 \end{cases} \tag{2.123}$$

2.5.6 Second Order Linear ODE with non-Constant Coefficients

Let $f_1(x)$, $f_2(x)$ be two linear independent solutions of the homogeneous problem $h(x) = 0$; $W[f_1, f_2] = \det\begin{pmatrix} f_1 & f_2 \\ f_1^{(1)} & f_2^{(1)} \end{pmatrix}$.

$$\begin{aligned} &f^{(2)}(x) + p(x)f^{(1)}(x) + q(x)f(x) = h(x) \iff \\ &f(x) = c_1 f_1(x) + c_2 f_2(x) - f_1(x)\int_{x_0}^x \mathrm{d}s\, \frac{h(s)f_2(s)}{W[f_1,f_2](s)} + f_2(x)\int_{x_0}^x \mathrm{d}s\, \frac{h(s)f_1(s)}{W[f_1,f_2](s)} \end{aligned} \tag{2.124}$$

2.6 Hilbert Spaces and Transformations

2.6.1 Legendre Transform

Let $\mathbf{x} \in \mathbb{R}^N$; $f : \mathbb{R}^N \to \mathbb{R}$; $(\mathrm{D}^2 f)(\mathbf{x})$ the Hessian matrix of elements $\frac{\partial^2 f}{\partial x_i \partial x_j}$; $y_i = \frac{\partial f}{\partial x_i}(\mathbf{x})$; $\mathbf{y} \in \mathbb{R}^N$; $x_i = \phi_i(\mathbf{y})$. We define the Legendre transform $f_*(\mathbf{y})$ of $f(\mathbf{x})$ by

$$f_*(\mathbf{y}) = \sum_{i=1}^N y_i \phi_i(\mathbf{y}) - f(\phi(\mathbf{y})) \tag{2.125}$$

$$f_*(\mathbf{y}) = \begin{cases} \sup_{\mathbf{x}\in\mathbb{R}^N}(\mathbf{y}\cdot\mathbf{x} - f(\mathbf{x})), & \mathrm{D}^2 f(\mathbf{x}) > 0\,\forall\mathbf{x} \in \Omega \subset \mathbb{R}^N \\ \inf_{\mathbf{x}\in\mathbb{R}^N}(\mathbf{y}\cdot\mathbf{x} - f(\mathbf{x})), & \mathrm{D}^2 f(\mathbf{x}) < 0\,\forall\mathbf{x} \in \Omega \subset \mathbb{R}^N \end{cases} \tag{2.126}$$

2.6.2 Laplace Transform

Let $f : [0, \infty[\to \mathbb{C}$.

$$\mathscr{L}\left(f(x)\right)(s) = F(s) = \int_0^\infty \mathrm{d}x\, f(x) \mathrm{e}^{-sx} \tag{2.127}$$

$$\mathscr{L}\left(\mathrm{e}^{ax} f(x)\right) = F(s-a) \tag{2.128}$$

$$\mathscr{L}\left(x^n f(x)\right) = (-1)^n F^{(n)}(s) \tag{2.129}$$

$$\mathscr{L}\left(f^{(n)}(x)\right) = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) \tag{2.130}$$

$$(f * g)(x) = \int_0^x \mathrm{d}y\, f(x-y) g(y) \Longrightarrow \mathscr{L}(f*g) = \mathscr{L}(f)\mathscr{L}(g) \tag{2.131}$$

2.6.3 Fourier Transform

Let $f : \mathbb{R}^N \to \mathbb{C}$ so that $\int_{\mathbb{R}^N} \mathrm{d}^N\mathbf{x}\, |f(\mathbf{x})|^2 < \infty$.

$$\mathscr{F}\left(f(\mathbf{x})\right)(\mathbf{k}) = \hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N\mathbf{x}\, f(\mathbf{x}) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \tag{2.132}$$

$$\mathscr{F}^{-1}\left(\hat{f}(\mathbf{k})\right)(\mathbf{x}) = f(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N\mathbf{x}\, \hat{f}(\mathbf{k}) \mathrm{e}^{i\mathbf{x}\cdot\mathbf{k}} \tag{2.133}$$

$$\mathscr{F}\left(f(\mathbf{x}+\mathbf{h})\right)(\mathbf{k}) = \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{h}} \mathscr{F}\left(f(\mathbf{x})\right)(\mathbf{k}) \tag{2.134}$$

$$\mathscr{F}\left(f\left(a\mathbf{x}\right)\right)(\mathbf{k}) = \tfrac{1}{|a|^N} \mathscr{F}\left(f(\mathbf{x})\right)(\mathbf{k}) \tag{2.135}$$

$$\mathscr{F}\left(f^{(n)}\left(\mathbf{x}\right)\right)(\mathbf{k}) = (i\mathbf{k})^n\, \mathscr{F}\left(f(\mathbf{x})\right)(\mathbf{k}) \tag{2.136}$$

$$\mathscr{F}\left(\mathbf{x}^n f\left(\mathbf{x}\right)\right)(\mathbf{k}) = (i)^n\, \mathscr{F}^{(n)}\left(f(\mathbf{x})\right)(\mathbf{k}) \tag{2.137}$$

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^N} \mathrm{d}^N\mathbf{y}\, f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) \Longrightarrow \mathscr{F}\left(f*g\right) = \mathscr{F}(f)\mathscr{F}(g) \tag{2.138}$$

$$\langle f | g \rangle = \langle \hat{f} | \hat{g} \rangle \tag{2.139}$$

$$\mathscr{F}\left(\mathrm{e}^{-\frac{1}{2}\langle \mathbf{x}|\boldsymbol{A}\mathbf{x}\rangle}\right)(\mathbf{k}) = \frac{1}{\left(\det(\boldsymbol{A})\right)^{1/2}} \mathrm{e}^{-\frac{1}{2}\left\langle \mathbf{k} \middle| \boldsymbol{A}^{-1}\mathbf{k}\right\rangle} \tag{2.140}$$

2.6.4 Bases of $L^2([a,b])$

Let $\langle f|g\rangle = \int_a^b \mathrm{d}x\, f(x)^\star g(x)$ be the scalar product of $L^2([a,b])$, then one has the following orthonormal bases $\{\mathcal{B}_i\}_{i=1}^4$ of $L^2([a,b])$. $\mathcal{B}_2$ leads to the usual Fourier sum.

$$\mathcal{B}_1 = \left\{ \frac{1}{\sqrt{b-a}} \exp\left(in \left(\frac{2\pi}{b-a} \right) x \right) \right\}_{n\in\mathbb{Z}} \tag{2.141}$$

$$\mathcal{B}_2 = \left\{ \frac{1}{\sqrt{b-a}}, \sqrt{\frac{2}{b-a}} \cos\left(n \left(\frac{2\pi}{b-a} \right) x \right), \sqrt{\frac{2}{b-a}} \sin\left(n \left(\frac{2\pi}{b-a} \right) x \right) \right\}_{n\in\mathbb{N}^*} \tag{2.142}$$

$$\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{b-a}}, \sqrt{\frac{2}{b-a}} \cos\left(n \left(\frac{\pi}{b-a} \right) (x-a) \right) \right\}_{n\in\mathbb{N}^*} \tag{2.143}$$

$$\mathcal{B}_4 = \left\{ \sqrt{\frac{2}{b-a}} \sin\left(n \left(\frac{\pi}{b-a} \right) (x-a) \right) \right\}_{n\in\mathbb{N}^*} \tag{2.144}$$

2.7 Special Functions

2.7.1 Bessel Functions $J_\nu(x)$, $Y_\nu(x)$, $H_\nu^{(\pm)}(x)$

•*First Kind Bessel Functions* $J_\nu(x)$
Let $\nu \geq 0$; $x \in]0, \infty[$; then $\{J_\nu, J_{-\nu}\}$ are two linear independent solutions.

$$x^2 f^{(2)}(x) + x f^{(1)}(x) + (x^2 - \nu^2) f(x) = 0 \iff J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+\nu}}{k!\,\Gamma(\nu + k + 1)} \tag{2.145}$$

•*Second Kind Bessel Functions (Neumann Functions)* $Y_\nu(x)$
$\{J_\nu, Y_\nu\}$ are two linear independent solutions of Bessel's equation, $\nu \notin \mathbb{N}$.

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)} \left(\cos(\nu\pi J_\nu(x)) - J_{-\nu}(x)\right) \tag{2.146}$$

If $\nu \in \mathbb{N}$

$$Y_\nu(x) = \frac{1}{\pi} \left(\frac{\partial}{\partial \nu} J_\nu(x) - (-1)^\nu \frac{\partial}{\partial \nu} J_{-\nu}(x) \right) \tag{2.147}$$

•*Third Kind Bessel Functions (Hankel's Functions)* $H_\nu^{(\pm)}(x)$
$\{H_\nu^{(+)}, H_\nu^{(-)}\}$ are two linear independent solutions of Bessel's equation.

$$H_\nu^{(\pm)}(x) = J_\nu(x) \pm i Y_\nu(x) \tag{2.148}$$

•*Asymptotic Behaviour of* $J_\nu(x)$, $Y_\nu(x)$

$$J_\nu(x) \overset{x \to +\infty}{\asymp} \sqrt{\frac{\pi}{2x}} \cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) + \frac{R_\nu(x)}{x^{3/2}}, \quad \sup_{x \geq 1} |R_\nu(x)| < \infty \tag{2.149}$$

$$Y_\nu(x) \overset{x \to +\infty}{\asymp} \sqrt{\frac{\pi}{2x}} \sin\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) + \frac{S_\nu(x)}{x^{3/2}}, \quad \sup_{x \geq 1} |S_\nu(x)| < \infty \tag{2.150}$$

2.7.2 Legendre Polynomials $P_n(x)$

$\sqrt{n + 1/2}\, P_n(x)$ form an orthonormal basis of $L^2(]-1, 1[, \mathrm{d}x)$.

$$(1 - x^2) f^{(2)}(x) - 2x f^{(1)}(x) + n(n+1) f(x) = 0 \iff P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n \tag{2.151}$$

$$\begin{array}{lll} P_0(x) = 1 & P_1(x) = x & P_2(x) = \frac{1}{2}(3x^2 - 1) \\ P_3(x) = \frac{1}{2}x(5x^2 - 3) & P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) & P_5(x) = \frac{1}{8}x(63x^4 - 70x^2 + 15) \end{array}$$

2.7.3 Associated Legendre Polynomials $P_n^m(x)$

$$(1 - x^2) f^{(2)}(x) - 2x f^{(1)}(x) + \left(n(n+1) - \frac{m^2}{1 - x^2} \right) f(x) = 0$$
$$\iff P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_n(x) \tag{2.152}$$

$$\begin{array}{lll} P_0^0(x) = 1 & P_1^0(x) = x & P_2^0(x) = \frac{1}{2}(3x^2 - 1) \\ P_0^1(x) = 0 & P_1^1(x) = -\sqrt{1 - x^2} & P_2^1(x) = -3x\sqrt{1 - x^2} \\ P_0^2(x) = 0 & P_1^2(x) = 0 & P_2^2(x) = 3(1 - x^2) \end{array}$$

2.7.4 Hermite Polynomials $H_n(x)$

$(2^n\pi^{1/2}n!)^{-1/2}H_n(x)$ form an orthonormal basis of $L^2\left(]-\infty,\infty[,\mathrm{e}^{-x^2}\mathrm{d}x\right)$.

$$f^{(2)}(x)-2xf^{(1)}(x)+2nf(x)=0 \iff H_n(x)=(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2} \tag{2.153}$$

$$\begin{array}{lll} H_0(x)=1 & H_1(x)=2x & H_2(x)=2(2x^2-1) \\ H_3(x)=4x(2x^2-3) & H_4(x)=4(4x^4-12x^2+3) & H_5(x)=8x(4x^4-20x^2+15) \end{array}$$

2.7.5 Generalized Laguerre Polynomials $L_n^a(x)$

$\sqrt{\frac{\Gamma(n+1)}{\Gamma(a+n+1)}}L_n^a(x)$ form an orthonormal basis of $L^2\left(]0,\infty[,x^a\mathrm{e}^{-x}\mathrm{d}x\right)$; $a>-1$.

$$\begin{aligned} &xf^{(2)}(x)+(a+1-x)f^{(1)}(x)+nf(x)=0 \\ &\iff L_n^a(x)=(-1)^n\mathrm{e}^x x^{-a}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x}x^{n+a}\right)=\sum_{j=0}^{n}(-1)^j\binom{n+a}{n-j}\frac{x^j}{j!} \end{aligned} \tag{2.154}$$

$$\begin{array}{lll} L_0^0(x)=1 & L_1^0(x)=1-x & L_2^0(x)=1-2x+\frac{1}{2}x^2 \\ L_0^1(x)=1 & L_1^1(x)=2-x & L_2^1(x)=3-3x+\frac{1}{2}x^2 \\ L_0^2(x)=1 & L_1^2(x)=3-x & L_2^2(x)=6-4x+\frac{1}{2}x^2 \end{array}$$

The Sonine polynomials $S_n(x)$ are defined by $S_n(x)=L_n^{d/2-1}$, where d is the dimension.

2.7.6 Chebyshev Polynomials $T_n(x)$

$\sqrt{\frac{\varepsilon_n}{\pi}}T_n(x)$ form an orthonormal basis of $L^2\left(]-1,1[,(1-x^2)^{-1/2}\mathrm{d}x\right)$; $\varepsilon_n=2-\delta_{n,0}$.

$$T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0 \iff T_n(x)=\cos\left(n\,\mathrm{acos}(x)\right)=\sum_{k=0}^{1}\frac{\left(x+i^{1+2k}\sqrt{1-x^2}\right)^n}{2} \tag{2.155}$$

$$\begin{array}{lll} T_0(x)=1 & T_1(x)=x & T_2(x)=2x^2-1 \\ T_3(x)=4x^2-3x & T_4(x)=8x^4-8x^2+1 & T_5(x)=16x^5-20x^3+5x \end{array}$$

2.7.7 Spherical Harmonics $Y_l^m(\theta,\varphi)$

$$Y_l^m(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}\sqrt{(l+m)!(l-m)!}\,\mathrm{e}^{im\varphi}P_l^m(\cos\theta)\,,\qquad l\geq 0 \tag{2.156}$$

$$Y_l^m(\theta,\varphi)=Y_{-(l+1)}^m(\theta,\varphi)\,,\qquad l\leq -1 \tag{2.157}$$

$$Y_0^0(\theta,\varphi)=\frac{1}{\sqrt{4\pi}} \tag{2.158}$$

$$Y_1^0(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta \tag{2.159}$$

$$Y_1^{\pm1}(\theta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\pm i\varphi} \tag{2.160}$$

$$Y_2^0(\theta,\varphi)=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) \tag{2.161}$$

$$Y_2^{\pm1}(\theta,\varphi)=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,\mathrm{e}^{\pm i\varphi} \tag{2.162}$$

$$Y_2^{\pm2}(\theta,\varphi)=\sqrt{\frac{15}{32\pi}}\sin^2\theta\,\mathrm{e}^{\pm 2i\varphi} \tag{2.163}$$

2.7.8 Radial Hydrogen Functions $R_{nl}(r)$

Let $a_0 = \frac{\hbar^2}{\mu e^2} = 0.529117$ Å; $C(n,l)$ be a numerical coefficient; $C(1,0) = 2$, $C(2,0) = \sqrt{2}$, $C(2,1) = \frac{1}{2\sqrt{6}}$.

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar l(l+1)}{2\mu r^2} - \frac{e^2}{r} - E_n\right) R_{nl} = 0$$

$$\iff R_{nl}(r) = C(n,l)\frac{1}{a_0^{3/2}}\left(\frac{r}{a_o}\right)^l \mathrm{e}^{-\frac{r}{a_0 n}} L^l_{n-l-1}\left(\frac{r}{2a_0}\right) \quad (2.164)$$

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\mathrm{e}^{-\frac{r}{a_0}} \quad (2.165)$$

$$R_{20}(r) = \frac{2}{(2a_0)^{3/2}}\left(1 - \frac{r}{2a_0}\right)\mathrm{e}^{-\frac{r}{2a_0}} \quad (2.166)$$

$$R_{21}(r) = \frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}\mathrm{e}^{-\frac{r}{2a_0}} \quad (2.167)$$

$$R_{30}(r) = \frac{2}{(3a_0)^{3/2}}\left(1 - \frac{2}{3}\frac{r}{a_0} + \frac{2}{27}\frac{r^2}{a_0^2}\right)\mathrm{e}^{-\frac{r}{3a_0}} \quad (2.168)$$

$$R_{31}(r) = \frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(1 - \frac{1}{6}\frac{r}{a_0}\right)\frac{r}{a_0}\mathrm{e}^{-\frac{r}{3a_0}} \quad (2.169)$$

$$R_{32}(r) = \frac{12}{81\sqrt{10}(3a_0)^{3/2}}\left(\frac{r}{a_0}\right)^2\mathrm{e}^{-\frac{r}{3a_0}} \quad (2.170)$$

2.7.9 Airy Function $A_i(x)$, $B_i(x)$; Asymptotic Behaviour

$$f^{(2)}(x) - xf(x) = 0 \iff f(x) = \sqrt{x}J_{1/3}\left(\tfrac{2}{3}x^{3/2}\right) \quad (2.171)$$

$A_i(+\infty) = A_i^{(1)}(+\infty) = 0$ $B_i(-\infty) = B_i^{(1)}(-\infty) = 0$

$A_i(x) \overset{x\to+\infty}{\asymp} \frac{1}{x^{1/4}}\exp\left(-\frac{2}{3}x^{3/2}\right)$ $B_i(x) \overset{x\to+\infty}{\asymp} \frac{1}{x^{1/4}}\exp\left(\frac{2}{3}x^{3/2}\right)$

$A_i(x) \overset{x\to-\infty}{\asymp} \frac{2}{(-x)^{1/4}}\cos\left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right)$ $B_i(x) \overset{x\to-\infty}{\asymp} \frac{1}{(-x)^{1/4}}\sin\left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right)$

2.7.10 Gamma Function $\Gamma(x)$

$$(x-1)! = \Gamma(x) = \int_0^\infty \mathrm{d}y\,\mathrm{e}^{-y}y^{x-1} \quad (2.172)$$

$\Gamma(x+1) = x\Gamma(x)$ $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

$\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!$ $\Gamma(1/2-n) = (-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!!}$

2.7.11 Stirling Formula

Let $z \in \mathbb{C}$. In general one uses $z! \sim \sqrt{2\pi}z^{z+1/2}\mathrm{e}^{-z}$.

$$\sqrt{2\pi}z^{z+1/2}\mathrm{e}^{-z}\mathrm{e}^{\frac{1}{12z+1}} < z! < \sqrt{2\pi}z^{z+1/2}\mathrm{e}^{-z}\mathrm{e}^{\frac{1}{12z}} \quad (2.173)$$

2.8 Statistics

2.8.1 Statistical Distributions

Distribution name	Distribution	$\langle X \rangle$	$(\Delta X)^2$
Binomial $b(n,p)$	$p(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Poisson $P(\lambda)$	$p(X=x)=\frac{\lambda^x}{x!}\mathrm{e}^{-\lambda}$	λ	λ
Geometric $G(p)$	$p(X=x)=p(1-p)^{x-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Pascal $Pa(n,x,p)$	$p(X=x)=\binom{n-1}{x-1}p^x(1-p)^{n-x}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Exponential $\exp(\lambda)$	$p(X=x)=\begin{cases}\lambda\mathrm{e}^{-\lambda x}, & x\geq 0\\ 0, & x<0\end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma $\Gamma(t,\lambda)$	$p(X=x)=\begin{cases}\frac{\lambda\mathrm{e}^{-\lambda x}(\lambda x)^{t-1}}{\Gamma(s)}, & x\geq 0\\ 0, & x<0\end{cases}$	$\frac{t}{\lambda}$	$\frac{t}{\lambda^2}$
Normal $N(\boldsymbol{\mu},\sigma)$	$p(X=\mathbf{x})=\frac{1}{(2\pi\sigma^2)^{N/2}}\mathrm{e}^{-\frac{(\mathbf{x}-\boldsymbol{\mu})^2}{2\sigma^2}}$	$\boldsymbol{\mu}$	σ^2

2.8.2 Inequalities (Markov, Jensen, Tchebychev)

Let $\phi:\mathbb{R}\to\mathbb{R}$ be convex.

$$p(|X|\geq a)\leq\frac{\langle |X|\rangle}{a};\qquad \phi\left(\langle X\rangle\right)\leq\langle\phi(X)\rangle\,;\qquad p\left(|X-\langle X\rangle|\geq a\right)\leq\frac{(\Delta X)^2}{a^2}$$

2.8.3 Limit Theorems

•*Central-Limit*
Let $\{X_i\}_{i=1}^n$ be independent and equally distributed; $\langle X_i\rangle=0\ \forall i$; $\rho=\frac{\langle |X|^3\rangle}{(\Delta X)^3}$; $\Phi(x)=\frac{1}{2\pi}\int_{-\infty}^{x}\mathrm{d}y\,\mathrm{e}^{-y^2/2}$.

$$\sup_{a\in\mathbb{R}}\left|p\left(\frac{1}{\Delta X\sqrt{n}}\sum_{i=1}^{n}X_i<a\right)-\Phi(a)\right|\leq\frac{3\rho}{\sqrt{n}}\qquad(2.174)$$

•*Moivre-Laplace*
Let $\{X_i\}_{i=1}^n$ be independent; $\langle X_i\rangle=\mu_i$.

$$\lim_{n\to\infty}p\left(a\leq\frac{\sum_{i=1}^{n}(X_i-\mu_i)}{\sqrt{\sum_{i=1}^{n}(\Delta X_i)^2}}\leq b\right)=\Phi(b)-\Phi(a)\qquad(2.175)$$

2.8.4 Wick's Theorem (Classical Case)

Let's consider a gaussian distribution; $(2k)!!=1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)=\frac{(2k)!}{k!2^k}$.

$$\langle\mathbf{x}_{k_1}\cdot\ldots\cdot\mathbf{x}_{k_n}\rangle=\begin{cases}\sum_{\text{pairs}}^{n!!}\left\langle\mathbf{x}_{k_{p_1}}\cdot\mathbf{x}_{k_{p_2}}\right\rangle\cdot\ldots\cdot\left\langle\mathbf{x}_{k_{p_{n-1}}}\cdot\mathbf{x}_{k_{p_n}}\right\rangle, & n\text{ even}\\ 0, & n\text{ odd}\end{cases}\qquad(2.176)$$

3 Physics

3.1 Motion in Curvilinear Coordinates

3.1.1 Cylindrical Coordinates

$$\begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \\ \dot{z} \end{pmatrix}; \quad \begin{pmatrix} a_r \\ a_\varphi \\ a_z \end{pmatrix} = \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ r\ddot{\varphi} + 2\dot{r}\dot{\varphi} \\ \ddot{z} \end{pmatrix}$$

3.1.2 Spherical Coordinates

$$\begin{pmatrix} v_r \\ v_\theta \\ v_\varphi \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ r\sin(\theta)\dot{\varphi} \end{pmatrix}; \quad \begin{pmatrix} a_r \\ a_\theta \\ a_\varphi \end{pmatrix} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 - r\sin(\theta)^2\dot{\varphi}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin(\theta)\cos(\theta)\dot{\varphi}^2 \\ r\sin(\theta)\ddot{\varphi} + 2\sin(\theta)\dot{r}\dot{\varphi} + 2r\cos(\theta)\dot{\theta}\dot{\varphi} \end{pmatrix}$$

3.2 Classical Mechanics

3.2.1 Lagrange Equations with non Conservative Forces

Let $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ be the cartesian coordinates; $\mathbf{q} = (q_1, \ldots, q_N) \in \mathbb{R}^N$ the generalized coordinates; $C \leq N$ the number of holonom constraints; $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ the kinetic energy; $V(\mathbf{q}, t)$ the potential energy; $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, t)$ the Lagrangian; $\mathbf{F}(\mathbf{x}, t) \in \mathbb{R}^N$ the non conservative forces of the system.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \mathbf{F}\left(\mathbf{x}(\mathbf{q}), t\right) \cdot \frac{\partial \mathbf{x}(\mathbf{q})}{\partial q_i}, \qquad \forall\, i = 1, \ldots, N - C \tag{3.1}$$

3.3 Hydrodynamics

3.3.1 Navier-Stokes Equation

Let the flow be isotropic, incompressible, viscous, Newtonian; let μ be the dynamical viscosity; $\mathbf{f}$ the volumic forces.

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v}\right) = -\boldsymbol{\nabla} p + \rho\mathbf{f} + \mu\boldsymbol{\nabla}^2\mathbf{v} \tag{3.2}$$

When $\mu = 0$, (3.2) is called the Euler equation.

3.3.2 Bernoulli Equation

Let the flow be non viscous, stationary, of curl equal to zero; $C \in \mathbb{R}$.

$$p + \rho\frac{v_i^2}{2} + \rho f_i x_i = C \tag{3.3}$$

3.4 Classical Electrodynamics

3.4.1 Maxwell's Equations

Name or Description	Equation (SI)
1) Faraday's law	$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
2) Ampere's law	$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}$
3) Poisson equation	$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$
4) (Absence of magnetic monopoles)	$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$

3.4.2 Complementary Basic Relations

Name or description	Equation (SI)
Equation for the vector potential	$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}$
Equation for the scalar potential	$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$
Poisson Equation	$\nabla^2 f(\mathbf{x}) = g(\mathbf{x}) \iff f(\mathbf{x}) = -\frac{1}{4\pi}\int_\Omega \mathrm{d}^3\mathbf{y}\,\frac{g(\mathbf{y})}{\lvert\mathbf{x}-\mathbf{y}\rvert}$
Linear tensorial constitutive relations	$\mathbf{D} = \varepsilon\mathbf{E}; \qquad \mathbf{B} = \mu\mathbf{H}$
Electric Force	$\mathbf{F} = \chi\varepsilon_0 \nabla\left(\frac{1}{2}\mathbf{E}^2\right)\lvert\Omega\rvert$
Magnetic Force	$\mathbf{F} = \frac{\chi}{\mu_0}\nabla\left(\frac{1}{2}\mathbf{B}^2\right)\lvert\Omega\rvert$
Energy	$\mathrm{d}u = \mathbf{E}\cdot\mathrm{d}\mathbf{D} + \mathbf{H}\cdot\mathrm{d}\mathbf{B}$
	$\iff u(\mathbf{x},t) = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})$
Poynting's Vector	$\mathbf{S} = \mathbf{E}\times\mathbf{H}$
Poynting's Theorem	$\frac{\partial}{\partial t}u(\mathbf{x},t) + \nabla\cdot\mathbf{S}(\mathbf{x},t) = -\mathbf{E}\cdot\mathbf{j}$

3.4.3 Vacuum Electrostatics

Let $\Gamma \subset \mathbb{R}^3$ be a closed path; $\Sigma \subset \mathbb{R}^3$ a closed surface.

$$\nabla\times\mathbf{E} = 0 \iff \oint_\Gamma \mathrm{d}\gamma\,\mathbf{E} = 0 \tag{3.4}$$

$$\nabla\cdot\mathbf{D} = 0 \iff \oint_\Sigma \mathrm{d}\sigma\,\mathbf{D} = \sum_{i\in D_{\mathrm{int}}(\Sigma)} q_{\mathrm{i}} \tag{3.5}$$

$$\mathbf{E} = -\nabla V \iff \int_\Gamma \mathrm{d}\gamma\,\mathbf{E} = \int_a^b \mathrm{d}s\,\mathbf{E} = V_a - V_b \tag{3.6}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \iff V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_\Omega \mathrm{d}^3\mathbf{y}\,\frac{\rho(\mathbf{y})}{\lvert\mathbf{x}-\mathbf{y}\rvert} \tag{3.7}$$

$$\delta W = qV \iff \delta W = \mathbf{E}\cdot\mathrm{d}\mathbf{D} \tag{3.8}$$

3.4.4 Linear Theory of Conductors and Dielectrics

Let $\Omega \subset \mathbb{R}^3$ be a volume of measure $\lvert\Omega\rvert$.

$$V = \frac{Q}{C}; \qquad U = \frac{1}{2}CV^2; \qquad U = \frac{1}{2}\mathbf{E}^2 \tag{3.9}$$

$$\mathbf{F} = \chi\varepsilon_0\nabla\left(\frac{1}{2}\mathbf{E}^2\right)\lvert\Omega\rvert \tag{3.10}$$

$$\boldsymbol{\mu} = q\boldsymbol{\delta} \tag{3.11}$$

$$\mathbf{P} = \frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}\Omega}; \qquad \mathbf{P} = \chi\varepsilon_0\mathbf{E} \tag{3.12}$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \iff \mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} \tag{3.13}$$

3.4.5 Magnetostatics

$$\nabla\times\mathbf{H} = \mathbf{j} \iff \oint_\Gamma \mathrm{d}\gamma\,\mathbf{H} = \sum I_{\mathrm{int}} \tag{3.14}$$

$$\nabla\cdot\mathbf{B} = 0 \iff \oint_\Sigma \mathrm{d}\sigma\,\mathbf{B} = 0 \tag{3.15}$$

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{j} \iff \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_\Omega \mathrm{d}^3\mathbf{y}\,\frac{\mathbf{j}(\mathbf{y})}{\lvert\mathbf{x}-\mathbf{y}\rvert} \tag{3.16}$$

$$\mathrm{d}\mathbf{F} = I\mathrm{d}\mathbf{l}\times\mathbf{B} \iff \mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}I\mathrm{d}\mathbf{l}\times\frac{\hat{\mathbf{r}}}{r^2} \iff \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_\Omega \mathrm{d}^3\mathbf{y}\,\frac{\mathbf{j}(\mathbf{y})\times(\mathbf{x}-\mathbf{y})}{\lvert\mathbf{x}-\mathbf{y}\rvert^3} \tag{3.17}$$

$$\delta W = \mathbf{H}\cdot\mathrm{d}\mathbf{B} \tag{3.18}$$

3.4.6 Magnetism of Materials

$$\mathbf{F} = \frac{\chi}{\mu_0}\boldsymbol{\nabla}\left(\tfrac{1}{2}\mathbf{B}^2\right)|\Omega| \tag{3.19}$$

$$\mathbf{m} = I\mathbf{S}; \qquad \mathbf{M} = \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}\Omega}; \qquad \mathbf{M} = \chi\mathbf{H} \tag{3.20}$$

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) \iff \mathbf{B} = \mu_0\mu_r\mathbf{H} \tag{3.21}$$

3.4.7 Induction

$$\mathbf{j} = \sigma\mathbf{E} \iff V = RI \tag{3.22}$$

$$\Phi = \int_\Sigma \mathrm{d}\sigma\,\mathbf{B} \iff M_{ij} = \frac{\Phi_{ij}}{I_i}; \qquad L_i = \frac{\Phi_{ii}}{I_i} \tag{3.23}$$

$$U = \tfrac{1}{2}LI^2; \qquad U = \frac{\mathbf{B}^2}{2\mu_0} \tag{3.24}$$

$$\oint_\Gamma \mathrm{d}\gamma\,\mathbf{E} = \varepsilon \tag{3.25}$$

3.4.8 Kramers-Kronig Equations

Let $⨏_{\mathbb{R}}\mathrm{d}\omega\,\frac{f(\omega)}{\omega-\omega_0} = \lim_{R\to\infty}\lim_{\varepsilon\to 0}\left(\int_{-R}^{\omega_0-\varepsilon}\mathrm{d}\omega\,\frac{f(\omega)}{\omega-\omega_0} + \int_{\omega_0+\varepsilon}^{R}\mathrm{d}\omega\,\frac{f(\omega)}{\omega-\omega_0}\right)$; $\tilde\varepsilon(\omega) = \tilde\varepsilon_1(\omega) + i\tilde\varepsilon_2(\omega)$.

$$\tilde\varepsilon_1(\omega_0) - \varepsilon_0 = \frac{1}{\pi}⨏_{\mathbb{R}}\mathrm{d}\omega\,\frac{\tilde\varepsilon_2(\omega)}{\omega-\omega_0} \tag{3.26}$$

$$\tilde\varepsilon_2(\omega_0) = -\frac{1}{\pi}⨏_{\mathbb{R}}\mathrm{d}\omega\,\frac{\tilde\varepsilon_1(\omega)-\varepsilon_0}{\omega-\omega_0} \tag{3.27}$$

3.5 Special Relativity

3.5.1 Lorentz Transformation

Let $\boldsymbol{\beta} = \boldsymbol{v}/c$; $\gamma = \left(1-\boldsymbol{\beta}^2\right)^{-1/2}$; $\mathbf{x} = \left(ct, x^1, x^2, x^3\right)$; $x'_\mu = \Lambda^\mu{}_\nu x^\nu$; $\Lambda^\mu{}_\nu = g^{\mu\rho}\Lambda_{\rho\nu}$.

$$\left(\Lambda_{\mu\nu}\right)^4_{\mu,\nu=1} = \left(\frac{\partial x'_\mu}{\partial x_\nu}\right)^4_{\mu,\nu=1} = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{(\gamma-1)\beta_1^2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\frac{(\gamma-1)\beta_2^2}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\frac{(\gamma-1)\beta_3^2}{\beta^2} \end{pmatrix} \tag{3.28}$$

$$\left(g_{\mu\nu}\right)^4_{\mu,\nu=1} = \left(g^{\mu\nu}\right)^4_{\mu,\nu=1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{3.29}$$

If $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ are the parallel and perpendicular components by respect to $\mathbf{v}_1$, then the Lorentz law of composition of the speed $\mathbf{v}_1$ and $\mathbf{v}_2$ gives

$$\mathbf{v}_\parallel = \frac{\mathbf{v}_1 + \mathbf{v}_{2,\parallel}}{1+\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}}; \qquad \mathbf{v}_\perp = \frac{\sqrt{1-\frac{\mathbf{v}_1^2}{c^2}}}{1+\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}}\mathbf{v}_{2,\perp}.$$

3.5.2 Covariance, Contravariance, Invariance

•*Contravariant Field*

$$b'^\mu(\mathbf{x}') = \Lambda^\mu{}_\nu b^\nu(\mathbf{x}); \qquad b^\mu = g^{\mu\nu}b_\nu \tag{3.30}$$

•*Covariant Field*

$$b'_\mu(\mathbf{x}') = \left(\Lambda^{-1}\right)^\nu{}_\mu\, b_\mu(\mathbf{x})\,; \qquad b_\mu = g_{\mu\nu} b^\nu \tag{3.31}$$

•*(Pseudo)-Tensor Contravariant of Order p and Covariant of Order q*

$$b'^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} = C(\Lambda)\Lambda^{\alpha_1}{}_{\gamma_1} \ldots \Lambda^{\alpha_p}{}_{\gamma_p} (\Lambda^{-1})^{\delta_1}{}_{\beta_1} \ldots (\Lambda^{-1})^{\delta_q}{}_{\beta_q} b^{\gamma_1 \ldots \gamma_p}{}_{\delta_1 \ldots \delta_q} \tag{3.32}$$

- $C(\Lambda) = 1\, \forall\Lambda \Longrightarrow b$ is a tensor.
- $C(\Lambda) \neq 1\, \forall\Lambda \Longrightarrow b$ is a pseudo-tensor.
- $C(\Lambda) = \text{sign}(\Lambda^0{}_0)\, \forall\Lambda \Longrightarrow b$ is a pseudochron tensor.
- $C(\Lambda) = (\det(\Lambda))^{-1}\ \forall\Lambda \Longrightarrow b$ is a density tensor.

•*Invariance*

If a^μ is contravariant, b_μ covariant, then $a^\mu b_\nu$ is invariant under the Lorenz group.

$$a'^\mu(\mathbf{x}')b'_\mu(\mathbf{x}') = a^\mu(\mathbf{x})b_\mu(\mathbf{x}) \quad \forall\Lambda \tag{3.33}$$

The norm of the energy-momentum quadrivector $\mathbf{E} = (E/c, \mathbf{p})$ is an invariant.

$$\langle \mathbf{E}'|g|\mathbf{E}'\rangle(\mathbf{x}') = \langle \mathbf{E}|g|\mathbf{E}\rangle(\mathbf{x}) \tag{3.34}$$

3.5.3 Electrodynamics

Let $\rho(\mathbf{x}, t) = q\delta(\mathbf{x} - \mathbf{r}_q(t))$; $\mathbf{j}(\mathbf{x}, t) = q\mathbf{v}_q(t)\delta(\mathbf{x} - \mathbf{r}_q(t))$; τ the proper time defined by $\frac{d\tau}{dt} = 1/\gamma(t)$; $\mathbf{u}(\tau) = (\gamma, \gamma\mathbf{v}(t(\tau))/c)$ the proper speed.

Name or Description	Equation (SI)
Current quadrivector	$(J^\mu(\mathbf{x}))_{\mu=1}^4 = (\rho(\mathbf{x}, t), \mathbf{j}(\mathbf{x}, t)/c)$
Potential quadrivector	$(\mathcal{A}^\mu(\mathbf{x}, t))_{\mu=1}^4 = (\psi(\mathbf{x}, t), c\mathbf{A}(\mathbf{x}, t))$
Gauge transform	$\tilde{\mathcal{A}}_\mu = \mathcal{A}_\mu + \partial_\mu\chi(\mathbf{x}, t)$
Lorentz gauge	$\partial_\mu\mathcal{A}^\mu = 0$
Electromagnetic field tensor	$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu = -\mathcal{F}_{\nu\mu}$
Maxwell equations	$\partial^\mu\mathcal{F}_{\mu\nu} = J_\nu/\varepsilon_0$
	$\partial_\lambda\mathcal{F}_{\mu\nu} + \partial_\mu\mathcal{F}_{\nu\lambda} + \partial_\nu\mathcal{F}_{\lambda\mu} = 0\ \ \forall\lambda\mu\nu = 0, \ldots, 4$
Force quadrivector	$F^\mu = \mathcal{F}^{\mu\nu}u_\nu = \gamma(\mathbf{E}\cdot\mathbf{u}/c, \mathbf{E} + \mathbf{u}\times\mathbf{B})$
Impulsion quadrivector	$p^\mu = mcu^\mu$
Equations of motion	$\frac{dp^\mu}{d\tau} = qF^\mu$
Energy-impulsion tensor	$T^{\mu\nu} = \varepsilon_0(\mathcal{F}^\mu{}_\sigma\mathcal{F}^{\sigma\nu} - g^{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}/4)$
Force density quadrivector	$f^\mu = \mathcal{F}^{\mu\nu}J_\nu = (\mathbf{E}\cdot\mathbf{j}/c, \rho(\mathbf{E} + \mathbf{j}\times\mathbf{B}))$
Conservation laws	$\partial_\mu T^{\mu\nu} = -f^\nu$

3.6 Thermodynamics

3.6.1 Thermodynamic Functions

$S = S(U, \{X_j\}, V)$	$\frac{1}{T} = \frac{\partial S}{\partial U}$; $F_j = \frac{\partial S}{\partial X_j}$; $\frac{p}{T} = \frac{\partial S}{\partial V}$	$S = \frac{U}{T} + \sum_j F_j X_j + \frac{p}{T}V$
$U = U(S, \{X_j\}, V)$	$T = \frac{\partial U}{\partial S}$; $P_j = \frac{\partial U}{\partial X_j}$; $-p = \frac{\partial U}{\partial V}$	$U = TS + \sum_j P_j X_j - pV$
$F = F(T, \{X_j\}, V) = U - TS$	$S = -\frac{\partial F}{\partial T}$; $P_j = \frac{\partial F}{\partial X_j}$; $-p = \frac{\partial F}{\partial V}$	$F = \sum_j P_j X_j - pV$
$H = H(S, \{X_j\}, p) = U + pV$	$T = \frac{\partial H}{\partial S}$; $P_j = \frac{\partial H}{\partial X_j}$; $V = \frac{\partial H}{\partial p}$	$H = TS + \sum_j P_j X_j - pV$
$G = U - TS + pV$	$S = -\frac{\partial G}{\partial T}$; $P_j = \frac{\partial G}{\partial X_j}$; $V = \frac{\partial G}{\partial p}$	$G = \sum_j P_j X_j - pV$
$\Phi = U - TS - \sum_j \mu_j N_j$	$s = \frac{\partial p}{\partial T}$; $n_j = \frac{\partial p}{\partial \mu_j}$	$\Phi = -pV$

$$C_\xi = \left.\frac{\partial U}{\partial T}\right|_\xi\,; \qquad \chi = \left.\frac{\partial^2 f}{\partial h^2}(h, T)\right|_{h=0}\,; \qquad m = \frac{\partial f}{\partial h}(h, T)\,; \qquad \mathcal{L}_{1\to 2} = \frac{H_2}{m_2} - \frac{H_1}{m_1}$$

3.7 Classical Statistical Physics

3.7.1 Statistical Ensembles

See sections 3.10.1-3.10.3 page 28.

3.7.2 k-Points Correlation Function

Let $\Lambda \subset \mathbb{R}^d$; indistinguishable particles; $\mathbf{x}_\alpha = (\mathbf{p}_\alpha, \mathbf{q}_\alpha)$; $\mathbf{q}_\alpha \in \Lambda$; $\mathbf{p}_\alpha \in \mathbb{R}^d$; $\mathrm{d}^N\omega(\mathbf{x}) = \frac{\mathrm{d}^N\mathbf{x}}{N!h^{dN}}$; let

$$A(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \frac{1}{k!} \sum_{\substack{1\le\alpha_1\le N \\ \vdots \\ 1\le\alpha_k\le N}} A^{(k)}(\mathbf{x}_{\alpha_1}, \ldots, \mathbf{x}_{\alpha_k})$$

be a k-points observable; let $N = k+m$; then the k-points correlation function $\rho^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is given by

$$\rho^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \sum_{m=0}^{\infty} \int \mathrm{d}^m\omega(\mathbf{y}_1, \ldots, \mathbf{y}_m)\ \rho_{k+m}(\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_m) \quad (3.35)$$

$$\langle A \rangle_\rho = \int \mathrm{d}^k\omega(\mathbf{x})\, \rho^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k)\ A^{(k)}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \quad (3.36)$$

3.7.3 Fokker-Plank Equation

Let the stochastic process be a Markov weakly stationary process defined by $p(\mathbf{x}_0|\mathbf{x}, t)$; let $\mathbf{x} = (x_1, \ldots, x_n)$ be the n variables of the system; let $\alpha > 1$; $\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\,(\mathbf{x}-\mathbf{x}_0)_i\, p(\mathbf{x}_0|\mathbf{x}, t) = a_i(\mathbf{x}_0)\, t + O(t^\alpha)$; $\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{x}\,(\mathbf{x}-\mathbf{x}_0)_i(\mathbf{x}-\mathbf{x}_0)_j\, p(\mathbf{x}_0|\mathbf{x}, t) = b_{ij}(\mathbf{x}_0)\, t + O(t^\alpha)$.

$$\frac{\partial}{\partial t} p(\mathbf{x}_0|\mathbf{y}, t) = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left(a_i(\mathbf{y}) p(\mathbf{x}_0|\mathbf{y}, t)\right) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j} \left(b_{ij}(\mathbf{y}) p(\mathbf{x}_0|\mathbf{y}, t)\right) \quad (3.37)$$

3.7.4 Master Equations

Let the stochastic process be a discrete Markov weakly stationary process; $\mathcal{W}(n_1|n_2) = \lim_{t\to 0} \frac{1}{t} p(n_1|n_2, t)$; $P(n, t) = P(n'|n, t)$ the conditional probability of state n at time t granted that the state was n' at time 0, Ω the space phase.

$$\frac{d}{dt} p(n, t) = \sum_{n'\in\Omega} \left(p(n', t)\mathcal{W}(n'|n) - p(n, t)\mathcal{W}(n|n')\right) \quad (3.38)$$

3.8 Quantum Mechanics

3.8.1 Stationary Perturbation Theory

Let $H = H_0 + W$; $H_0 \,|\varphi_n\rangle = E_n^0 \,|\varphi_n\rangle$.

$$E_n = E_n^0 + \langle \varphi_n | W \varphi_n \rangle + \sum_{i\neq n} \frac{|\langle \varphi_i | W | \varphi_n \rangle|^2}{E_n^0 - E_i^0} + \ldots \quad (3.39)$$

$$\left|\varphi_n^{W\neq 0}\right\rangle = |\varphi_n\rangle + \sum_{i\neq n} \frac{\langle \varphi_i | W | \varphi_n \rangle}{E_n^0 - E_i^0} \,|\varphi_i\rangle + \ldots \quad (3.40)$$

3.8.2 Non Stationary Perturbation Theory

Let $H(t) = H_0 + \lambda W(t)$; $\langle x|\psi_t\rangle = U(t)\,\langle x|\psi_0\rangle$; $U_0(t) = \mathrm{e}^{-itH_0/\hbar}$; $W_I(t) = U_0^\dagger(t)W(t)U_0(t)$.

$$U(t) = U_0(t)\left(\mathbb{1} + \frac{\lambda}{i\hbar}\int_0^t \mathrm{d}s_1\, W_I(s_1) + \left(\frac{\lambda}{i\hbar}\right)^2 \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2\, W_I(s_1)W_I(s_2) + \dots \right.$$
$$\left. + \left(\frac{\lambda}{i\hbar}\right)^n \int_0^t \mathrm{d}s_n\, W_I(s_n) \prod_{k=1}^{n-1} \int_0^{s_{k+1}} \mathrm{d}s_k\, W_I(s_k) + \dots \right) \quad (3.41)$$

3.8.3 Fermi's Golden Rule

Let $H = H_0 + W(t)$; $H_0\,|\mathrm{in}\rangle = E_{\mathrm{in}}\,|\mathrm{in}\rangle$; $H_0\,|\mathrm{end}\rangle = E_{\mathrm{end}}\,|\mathrm{end}\rangle$; $\langle \mathrm{in}|\mathrm{end}\rangle = 0$; let $\{|\nu\rangle\}_\nu$ be an orthonormal base; $H_0\,|\nu\rangle = E_\nu\,|\nu\rangle$.

$$\frac{dp(\mathrm{in}\to\mathrm{end})}{dt} = \frac{2\pi}{\hbar}\,\delta(E_{\mathrm{in}} - E_{\mathrm{end}}) \left|\lim_{\eta\to 0} A(\eta)\right|^2 \quad (3.42)$$

$$A(\eta) = \begin{cases} \langle \mathrm{end}|W|\mathrm{in}\rangle\,, & \text{first order} \\ \displaystyle\sum_\nu \frac{\langle \mathrm{end}|W|\nu\rangle\,\langle \nu|W|\mathrm{in}\rangle}{E_\nu - E_{\mathrm{in}} - i\eta}, & \text{second order} \end{cases} \quad (3.43)$$

3.8.4 Harmonic Oscillator

Let $\{|n\rangle\}_{n=0}^\infty$ be the eigenstates of the hamiltonian H with eigenvalues $\{E_n\}_{n=0}^\infty$, so that $H\,|n\rangle = E_n\,|n\rangle$; $H_n(x)$ the n^{th} Hermite polynomial.

$$H = \frac{\hat{\mathbf{p}}^2}{2m} + \tfrac{1}{2}m\omega^2\hat{\mathbf{q}}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right) \quad (3.44)$$

$\hat{a} = \frac{\hat{\mathbf{p}} - im\omega\hat{\mathbf{q}}}{\sqrt{2\hbar m\omega}}$

$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{a}\,|0\rangle = 0$

$\hat{a}^\dagger = \frac{\hat{\mathbf{p}} + im\omega\hat{\mathbf{q}}}{\sqrt{2\hbar m\omega}}$

$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle$

$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}$

$\varphi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(2\frac{m\omega}{\hbar} - 1\right)\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}$

$\varphi_n(x) = \langle x|n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$

$\varphi_1(x) = \left(\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right)^{1/4} x\,\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2}$

3.8.5 Hydrogen Atom

Let $r = |\mathbf{r}|$; $\mathbf{r}\in\mathbb{R}^3$; $\mu = \frac{m_e m_p}{m_e + m_p}$; $e^2 = \frac{q^2}{4\pi\varepsilon_0}$; $\mu_B = \frac{q\hbar}{2m_e}$. $R_{nl}(r)$ and $Y_l^m(\theta,\varphi)$ are defined in the sections 2.7.7 and 2.7.8. The complete set of commutating observables is $\{H, \mathbf{L}^2, L_3\}$.

$$\left(-\frac{\hbar^2}{2\mu}\boldsymbol{\nabla}^2 - \frac{e^2}{r}\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}) \quad (3.45)$$

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_l^m(\theta,\varphi) \quad (3.46)$$

$$H\psi_{nlm} = E_n\psi_{nlm} \qquad n = 1,2,\dots,\infty \quad (3.47)$$

$$\mathbf{L}^2 Y_l^m(\theta,\varphi) = \hbar^2 l(l+1)Y_l^m(\theta,\varphi) \qquad l = 0,1,\dots,n-1 \quad (3.48)$$

$$L_3 Y_l^m(\theta,\varphi) = \hbar m Y_l^m(\theta,\varphi) \qquad m = -l,-l+1,\dots,l-1,l \quad (3.49)$$

$$E_n = -\frac{E_I}{n^2} \tag{3.50}$$

$$E_I = \frac{\alpha^2 m_e c^2}{2} = 13.60580 \text{ eV}, \qquad \alpha = \frac{e^2}{\hbar c} \tag{3.51}$$

$$\boldsymbol{\nabla}^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\mathbf{L}^2}{\hbar^2 r^2} \tag{3.52}$$

$$L_\pm = L_1 \pm i L_2 \tag{3.53}$$

$$L_\pm Y_l^m(\theta,\varphi) = \hbar\sqrt{l(l+1) - m(m\pm 1)}\, Y_l^{m\pm 1}(\theta,\varphi) \tag{3.54}$$

$$\mathbf{L}^2 = \tfrac{1}{2}(L_+ L_- + L_- L_+) + L_3^2 \tag{3.55}$$

$$L_\pm L_\mp = \mathbf{L}^2 \pm \hbar L_3 - L_3^2 \tag{3.56}$$

3.8.6 Pauli's Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.8.7 Commutation and Anticommutation Relations

Let $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$, $\hat{\mathbf{L}} = \hat{\mathbf{q}} \times \hat{\mathbf{p}}$ be the position, impulsion, orbital kinetic momentum operators; $\hat{a}$, $\hat{a}^\dagger$ the annihilation, creation operators; $\circlearrowleft$ denotes a cyclic permutation.

$$\begin{array}{lll}
[\hat{\mathbf{q}}_\mathbf{i}, \hat{\mathbf{p}}_\mathbf{j}] = i\hbar\delta_{ij}; & [L_1, L_2] = i\hbar L_3 \circlearrowleft; & [\hat{\mathbf{L}}^2, \hat{\mathbf{L}}] = [\hat{\mathbf{L}}^2, L_3] = 0 \\
[L_3, L_\pm] = \pm\hbar L_\pm; & [L_+, L_-] = 2\hbar L_3; & [\hat{\mathbf{L}}^2, L_\pm] = 0 \\
\left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1}; & \left[\hat{a}^\dagger\hat{a}, \hat{a}\right] = -\hat{a}; & \left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger \\
[\sigma_x, \sigma_y] = 2i\sigma_z \circlearrowleft; & \{\sigma_x, \sigma_y\} = 0 \circlearrowleft &
\end{array}$$

$$\begin{array}{ll}
[AB, C] = A[B,C] + [A,C]B; & [A, BC] = B[A,C] + [A,B]C \\
[AB, C] = A\{B,C\} - \{A,C\}B; & [A, BC] = \{A,B\}C - B\{A,C\}
\end{array}$$

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} \tag{3.57}$$

$$\mathrm{e}^{A+B} = \lim_{N\to\infty}\left(\mathrm{e}^{\frac{A}{N}}\mathrm{e}^{\frac{B}{N}}\right)^N \tag{3.58}$$

$$\mathrm{e}^{\lambda A} B \mathrm{e}^{-\lambda A} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\Omega_n(A,B), \qquad \Omega_n(A,B) = \begin{cases} B, & n = 0 \\ [A, \Omega_{n-1}(A,B)], & n \geq 1 \end{cases} \tag{3.59}$$

3.9 Feynman Path Integral

3.9.1 Feynman Path Integral

Let $\mathcal{L}(\mathbf{x}(s), \dot{\mathbf{x}}(s))$ be the Lagrangian of the system; d its dimension.

$$\langle \mathbf{x}|U(t,t_0)|\mathbf{x}_0\rangle = \int_{\mathbf{x}_0,t_0}^{\mathbf{x},t} \mathrm{d}[\mathbf{x}(\cdot)]\, \exp\left(\frac{i}{\hbar}S(\mathbf{x}(\cdot))\right) \tag{3.60}$$

$$\int_{\mathbf{x}_0,t_0}^{\mathbf{x},t} \mathrm{d}[\mathbf{x}(\cdot)] = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\frac{t-t_0}{N+1}}\right)^{d\frac{N+1}{2}} \int_{\mathbb{R}^{dN}} \mathrm{d}^d\mathbf{x}_1 \cdot \ldots \cdot \mathrm{d}^d\mathbf{x}_N \tag{3.61}$$

$$S(\mathbf{x}(\cdot)) = \int_{t_0}^{t} \mathrm{d}s\, \mathcal{L}(\mathbf{x}(s), \dot{\mathbf{x}}(s)) \tag{3.62}$$

$$\langle \mathbf{x}|\psi_t\rangle = \int_{\mathbb{R}^d} \mathrm{d}^d\mathbf{x}_0\, \langle \mathbf{x}|U(t,t_0)|\mathbf{x}_0\rangle \langle \mathbf{x}_0|\psi_0\rangle \tag{3.63}$$

3.9.2 Van Vleck's Formula

Let $\mathbf{x}_c(\cdot)$ be the classical path with boundary conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t) = \mathbf{x}$; $\nu\left(\mathbf{x}_c(\cdot)\right)$ the number of times that $\det\left(\left\{\frac{\partial^2 S(\mathbf{x}_c(s))}{\partial x_{0,i}\partial x_j}\right\}_{i,j=1}^{3}\right) = \infty$, $s \in [0,t]$.

$$\langle \mathbf{x}|U(t,t_0)|\mathbf{x}_0\rangle \simeq \sum_{\mathbf{x}_c} \frac{1}{(2\pi i\hbar)^{3/2}} \left|\det\left(\left\{\frac{\partial^2 S(\mathbf{x}_c(s))}{\partial x_{0,i}\partial x_j}\right\}_{i,j=1}^{3}\right)\right|^{1/2} \mathrm{e}^{\frac{i}{\hbar}S(\mathbf{x}_c(\cdot))}\mathrm{e}^{-i\frac{\pi}{2}\nu(\mathbf{x}_c(\cdot))} \tag{3.64}$$

3.9.3 Exact Solutions

•*3D Free Particle:* $\mathcal{L} = \frac{1}{2}m\dot{\mathbf{x}}^2(s)$

$$\langle \mathbf{x}|U(t,t_0)|\mathbf{x}_0\rangle = \int_{\mathbf{x}_0,t_0}^{\mathbf{x},t} \mathrm{d}[\mathbf{x}(\cdot)]\, \exp\left(\frac{im}{2\hbar}\int_{t_0}^{t}\mathrm{d}s\,\dot{\mathbf{x}}^2(s)\right) = \left(\frac{m}{2\pi i\hbar(t-t_0)}\right)^{3/2} \mathrm{e}^{\frac{i}{\hbar}S(\mathbf{x}_c(\cdot))} \tag{3.65}$$

$$S\left(\mathbf{x}_c(\cdot)\right) = \frac{m}{2}\frac{\left(\mathbf{x}-\mathbf{x}_0\right)^2}{(t-t_0)} \tag{3.66}$$

$$\mathbf{x}_c(s) = \mathbf{x}_0 + \frac{s-t_0}{t-t_0}\left(\mathbf{x}-\mathbf{x}_0\right) \tag{3.67}$$

•*1D Linear Potential: Time Dependent Electrical Field:* $\mathcal{L} = \frac{1}{2}m\dot{x}^2(s) + eE(s)x(s)$

$$\langle x|U(t,0)|x_0\rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \mathrm{e}^{\frac{i}{\hbar}S(x_c(\cdot))} \tag{3.68}$$

$$x_c(s) = \frac{s}{t}(x-x_0) + x_0 + \frac{e}{m}\int_0^s \mathrm{d}u \int_0^u \mathrm{d}v\, E(v) - \frac{es}{mt}\int_0^t \mathrm{d}u \int_0^u \mathrm{d}v\, E(v) \tag{3.69}$$

•*1D Forced Harmonic Oscillator:* $\mathcal{L} = \frac{1}{2}m\dot{x}^2(s) - \frac{1}{2}m\omega^2 x^2(s) + F(s)x(s)$

$$\langle x|U(t,t_0)|x_0\rangle = \left(\frac{m}{2\pi i\hbar}\right)^{1/2}\left(\frac{\omega}{\sin\left(\omega(t-t_0)\right)}\right)^{1/2} \mathrm{e}^{\frac{i}{\hbar}S(x_c(\cdot))} \tag{3.70}$$

$$\begin{aligned} S\left(x_c(\cdot)\right) = \frac{m\omega}{2\sin\left(\omega(t-t_0)\right)} \Big\{ & (x^2+x_0^2)\cos\left(\omega(t-t_0)\right) - 2xx_0 \\ & + \frac{2x}{m\omega}\int_{t_0}^{t}\mathrm{d}s\, F(s)\sin\left(\omega(s-t_0)\right) + \frac{2x_0}{m\omega}\int_{t_0}^{t}\mathrm{d}s\, F(s)\sin\left(\omega(t-s)\right) \\ & - \frac{2}{m^2\omega^2}\int_{t_0}^{t}\mathrm{d}s \int_{t_0}^{s}\mathrm{d}u\, F(u)F(s)\sin\left(\omega(t-s)\right)\sin\left(\omega(u-t_0)\right) \Big\} \end{aligned} \tag{3.71}$$

$$\begin{aligned} x_c(s) = & \frac{\sin\left(\omega(s-t_0)\right)}{\sin\left(\omega(t-t_0)\right)}\left(x - x_0\cos\left(\omega(t-t_0)\right) - \frac{1}{m\omega}\int_{t_0}^{t}\mathrm{d}u\, F(u)\sin\left(\omega(t-u)\right)\right) \\ & + x_0\cos\left(\omega(s-t_0)\right) + \frac{1}{m\omega}\int_{t_0}^{s}\mathrm{d}u\, F(u)\sin\left(\omega(s-u)\right) \end{aligned} \tag{3.72}$$

3.9.4 Quantum Statistical Physics

$$Q = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right), \qquad t_0 = 0, t = \beta\hbar \tag{3.73}$$

$$Q = \int_{\mathbb{R}}\mathrm{d}x \int_{\mathbf{x},0}^{\mathbf{x},\beta\hbar}\mathrm{d}W_D\, \exp\left(-\frac{1}{\hbar}\int_0^{\beta\hbar}\mathrm{d}s\, V\left(\mathbf{x}(s)\right)\right) \tag{3.74}$$

$$\mathrm{d}W_D = \mathrm{d}\left[\mathbf{x}(\cdot)\right]\, \exp\left(-\frac{1}{4D}\int_0^{\beta\hbar}\mathrm{d}\sigma\, \left(\frac{\mathrm{d}\mathbf{x}(\sigma)}{\mathrm{d}\sigma}\right)^2\right), \qquad D = \frac{\hbar}{2m} \tag{3.75}$$

$$\langle \mathbf{x}(t_1)\cdot\mathbf{x}(t_2)\rangle = 2D\min\left(t_1,t_2\right) \tag{3.76}$$

3.10 Quantum Statistical Physics

In the classical case, one replaces $\mathrm{Tr}(\cdot)$ by $\sum_{N=0}^{\infty}\int_{\Gamma_{N,\Lambda}}\frac{\mathrm{d}^N\mathbf{x}}{N!h^{dN}}(\cdot)$ (the sum appears only in the grand canonical ensemble), with $\mathbf{x}=\{\hat{\mathbf{q}}_i,\hat{\mathbf{p}}_i\}_{i=1}^{N}$; $\hat{\mathbf{q}},\hat{\mathbf{p}}\in\mathbb{R}^d$; and operators by their eigenvalues.

3.10.1 Microcanonical Ensemble

Let P_E^{Δ} be the projector on the Hilbert subspace of energy $[E-\Delta,E]$.

$$\hat{\rho}^{\Delta}(E,N,\Lambda)=\frac{1}{\Omega^{\Delta}(E,N,\Lambda)}P_E^{\Delta} \tag{3.77}$$

$$\Omega^{\Delta}(E,N,\Lambda)=\mathrm{Tr}\left(P_E^{\Delta}\right) \tag{3.78}$$

$$S(E,N,\Lambda)=k_B\ln\left(\Omega^{\Delta}(E,N,\Lambda)\right)$$

3.10.2 Canonical Ensemble

$$\hat{\rho}(\beta,N,\Lambda)=\frac{\mathrm{e}^{-\beta H_{N,\Lambda}}}{Q(\beta,N,\Lambda)} \tag{3.79}$$

$$Q(\beta,N,\Lambda)=\mathrm{Tr}\left(\mathrm{e}^{-\beta H_{N,\Lambda}}\right) \tag{3.80}$$

$F(\beta,N,\Lambda)=-\frac{1}{\beta}\ln\left(Q(\beta,N,\Lambda)\right)$; $\quad S(\beta,N,\Lambda)=-k_B\beta^2\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\ln\left(Q(\beta,N,\Lambda)\right)\right)$

$\langle H_{N,\Lambda}\rangle=-\frac{\partial}{\partial\beta}\ln\left(Q(\beta,N,\Lambda)\right)$

3.10.3 Grand Canonical Ensemble

$$\hat{\rho}(\beta,\mu,\Lambda)=\frac{\mathrm{e}^{-\beta(H_\Lambda-\mu N)}}{\mathbb{Q}(\beta,\mu,\Lambda)} \tag{3.81}$$

$$\mathbb{Q}(\beta,\mu,\Lambda)=\mathrm{Tr}\left(\mathrm{e}^{-\beta(H_\Lambda-\mu N)}\right) \tag{3.82}$$

$p(\beta,\mu,\Lambda)=\frac{1}{\beta}\frac{1}{|\Lambda|}\ln\left(\mathbb{Q}(\beta,\mu,\Lambda)\right)$; $\quad S(\beta,\mu,\Lambda)=-k_B\beta^2\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\ln\left(\mathbb{Q}(\beta,\mu,\Lambda)\right)\right)$

$\langle N\rangle_{(\beta,\mu,\Lambda)}=\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\left(\mathbb{Q}(\beta,\mu,\Lambda)\right)$; $\quad \langle H\rangle=\mu\langle N\rangle-\frac{\partial}{\partial\beta}\ln\left(\mathbb{Q}(\beta,\mu,\Lambda)\right)$

$\Delta_N^2=\langle N^2\rangle-\langle N\rangle^2=\frac{1}{\beta}\frac{\partial}{\partial\mu}\langle N\rangle$; $\quad \Delta_H^2=\langle H^2\rangle-\langle H\rangle^2=-\frac{\partial}{\partial\beta}\langle H\rangle+\frac{\mu}{\beta}\frac{\partial}{\partial\mu}\langle H\rangle$

3.10.4 Quantum Linear Response Theory

•*Linear Response Function* $\chi_{AB}(t,t')$
Let $H(t)=H_0+H_I(t)$; $H_I(t)=-Bf(t)$; let A,B be observables; $\chi_{AB}(t,t')=\chi_{AB}(t-t')$; $\chi_{AB}(t)=0\ \forall t<0$.

$$\langle A\rangle(t)=\langle A\rangle_{\rho_0}+\int_0^t\mathrm{d}t'\,\chi_{AB}(t,t')f(t')+O(f^2) \tag{3.83}$$

•*Fluctuation-Dissipation Theorem*
Let $U_0(t)=\mathrm{e}^{-itH_0/\hbar}$; $B^0(t)=U_0^{\star}(t)BU_0(t)$; let $G_{AB}(t)=\frac{1}{2}\mathrm{Tr}\left(\hat{\rho}_0\left(AB^0(t)+B^0(t)A\right)\right)=\left\langle AB^0(t)\right\rangle$ be the temporal correlations of A and B; $\tilde{G}_{AB}(\omega)=\int_{\mathbb{R}}\mathrm{d}t\,\mathrm{e}^{i\omega t}G_{AB}(t)$.

$$\lim_{\varepsilon\to 0}\frac{1}{2i}\left(\tilde{\chi}_{BA}(\omega+i\varepsilon)-\tilde{\chi}^{\star}_{AB}(\omega+i\varepsilon)\right)=\frac{1}{\hbar}\,\mathrm{th}\left(\frac{\beta\hbar\omega}{2}\right)\tilde{G}_{AB}(\omega) \tag{3.84}$$

•*Kubo's Formula*

$$\chi_{AB}(t)=\int_0^{\beta}\mathrm{d}\tau\left\langle\frac{d}{ds}B^0(s)\Big|_{s=-i\hbar\tau}A^0(t)\right\rangle_{\rho_0} \tag{3.85}$$

3.11 Dynamical Systems and Fractals

3.11.1 Liapunov Exponents

Let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \simeq \left(\frac{\mathrm{D}\mathbf{F}}{\mathrm{D}\mathbf{x}}\right)_{\mathbf{x}^*} \cdot \mathbf{x}_0 = \mathbf{V}_t \cdot \mathbf{x}_0$; $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \simeq \left(\frac{\mathrm{D}\mathbf{f}^n}{\mathrm{D}\mathbf{x}}\right)(\mathbf{x}_0) = \frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{x}}(\mathbf{x}_n) \cdot \frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{x}}(\mathbf{x}_{n-1}) \cdot \ldots \cdot \frac{\mathrm{D}\mathbf{f}}{\mathrm{D}\mathbf{x}}(\mathbf{x}_0) = \mathbf{V}_n(\mathbf{x}_0)$. The dynamical system is chaotic in $\Omega = \left\{\mathbf{x}_0 \in \mathbb{R}^N | \lambda^1 > 0\right\}$.

$$\lambda^1 = \lim_{t\to\infty} \frac{1}{2t} \ln\left(\mathrm{Tr}\left(\mathbf{V}_t^t \cdot \mathbf{V}_t\right)\right) \tag{3.86}$$

$$\lambda^1 = \lim_{n\to\infty} \frac{1}{2n} \ln\left(\mathrm{Tr}\left(\mathbf{V}_n^t \cdot \mathbf{V}_n\right)\right) \tag{3.87}$$

3.11.2 Generalized Multifractal Dimension D_q

Let $N(\varepsilon)$ be the number of hyper-box C_i of spatial extension ε needed to cover the object; let $\mu_i = \mu(C_i) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, \chi_{C_i}(\mathbf{x}(t))$ be the measure of C_i; $q = 0$ corresponds to the usual fractal box-dimension of uniform weight.

$$D_q = \frac{1}{1-q} \lim_{\varepsilon\to 0} \frac{\ln\left(I(q,\varepsilon)\right)}{\ln\left(\frac{1}{\varepsilon}\right)} \tag{3.88}$$

$$I(q,\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q \tag{3.89}$$